

SECONDARY SCHOOL MATHEMATICS TEACHERS' PERCEPTIONS ABOUT INDUCTIVE REASONING AND THEIR INTERPRETATION IN TEACHING

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Abstract

Inductive reasoning is an essential tool for teaching mathematics to generate knowledge, solve problems, and make generalizations. However, little research has been done on inductive reasoning as it applies to teaching mathematical concepts in secondary school. Therefore, the study explores secondary school teachers' perceptions of inductive reasoning and interprets this mathematical reasoning type in teaching the quadratic equation. The data were collected from a questionnaire administered to 22 teachers and an interview conducted to expand their answers. Through the thematic analysis method, it was found that more than half the teachers perceived inductive reasoning as a process for moving from the particular to the general and as a way to acquire mathematical knowledge through questioning. Because teachers have little clarity about inductive phases and processes, they expressed confusion about teaching the quadratic equation inductively. Results indicate that secondary school teachers need professional learning experiences geared towards using inductive reasoning processes and tasks to form concepts and generalizations in mathematics.

Keywords: Perception, Inductive Reasoning, In-Service Mathematics Teachers, Secondary School

Abstrak

Penalaran induksi merupakan hal yang penting di pembelajaran matematika untuk membangun pengetahuan, pemecahan masalah, dan membuat generalisasi. Namun, baru sedikit penelitian yang telah dilakukan tentang penalaran induksi yang diterapkan di pembelajaran konsep matematika di sekolah menengah. Oleh karena itu, studi ini mengeksplorasi persepsi guru di jenjang sekolah menengah tentang penalaran induksi dan menjelaskan tipe penalaran matematika di pembelajaran persamaan kuadrat. Data dikumpulkan dari kuesioner terhadap 22 guru dan dari interview untuk mendapatkan jawaban lebih dalam. Melalui metode tematik analisis, ditemukan bahwa lebih dari separoh guru ini memahami bahwa penalaran induksi adalah suatu proses dari hal khusus ke umum dan sebagai cara untuk mendapatkan pengetahuan matematika melalui bertanya. Dikarenakan guru ini memiliki kejelasan tentang fase penalaran induksi dan prosesnya, mereka mengalami kebingungan tentang mengajarkan persamaan kuadrat secara induksi. Hasil penelitian ini menunjukkan bahwa guru di jenjang sekolah menengah ini membutuhkan pengalaman pembelajaran profesional tentang penggunaan proses penalaran induksi dan penugasan untuk membangun konsep dan generalisasi di matematika.

Kata kunci: Persepsi, Penalaran Induktif, Guru Matematika, Sekolah Menengah

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The processes of the knowledge discovery and construction of proofs in mathematics involve both inductive and deductive reasoning (Davydov, 1990; Lee, 2016). The first implies moving from the particular to the general, and the second moves from the general to the particular (Hodnik & Manfreda, 2015). This work focuses on inductive reasoning, although some teachers are accustomed to employing deductive reasoning to teach mathematics (Rott, 2021). Siswono, Hartono, and Kohar (2020) defined deductive reasoning as 'a process of deducing conclusions from known information (premise) based on formal logic rules, where the conclusions must come from information provided and do not need to

validate them with experiments' (p. 419). Inductive reasoning, on the other hand, is oriented to infer laws or general conclusions through observation and connection of particular instances (be they facts, premises, or particular cases of situations or a class of mathematical objects), and the conclusions can be verified by experimentation (Haverty, Koedinger, Klahr, & Alibali, 2000; Polya, 1957). According to Reid and Knipping (2010), three invariant characteristics of this type of reasoning are that it (a) comes from specific cases to conclude general rules, (b) uses what is known to conclude something unknown, and (c) is only probable but not certain.

Inductive reasoning has a core function in intellectual processes development for mathematics (Klauer & Phye, 2008; Mousa, 2017; Tomic, 1995). This type of reasoning is particularly important for learning mathematics in primary and secondary school, for two reasons. Firstly, it constitutes a teaching pathway for developing concepts and solving mathematics problems (e.g., Molnár, Greiff, & Csapó, 2013; Christou & Papageorgiou, 2007; Sriraman & Adrian, 2004). Inductive reasoning contributes to the formation of concepts because it 'lead[s] to detecting regularities, be it classes of objects represented by generic concepts, be it common structures among different objects, or be it schemata enabling the learners to identify the same basic idea within various contexts' (Klauer, 1996, p. 53). Secondly, it is one of the forms of reasoning that supports the process of generalizing numerical and figural patterns or mathematical objects (Cañadas, Castro, & Castro, 2008; 2009; Rivera & Becker, 2016).

The National Council of Teachers of Mathematics [NCTM] (2014) established that this form of mathematical reasoning must progress in students throughout each education level so that they can become more proficient in formulating conjectures and generalizations from specific cases. For that reason, secondary school teachers should develop and interpret the students' reasoning (AMTE, 2017; NCTM, 2020). However, several studies have reported that pre-service and in-service teachers have difficulties in solving generalization tasks from particular cases through inductive reasoning (Rivera & Becker, 2003; 2007; Sosa & Aparicio, 2020). In particular, they show difficulties associated with establishing a pattern and achieving the abstraction of the general when solving quadratic pattern tasks (Manfreda, Slapar, & Hodnik, 2012; Sosa, Aparicio, & Cabañas, 2019; 2020).

In this sense, knowing the type of perceptions that teachers have about inductive reasoning and how they promote it in teaching is essential to address these difficulties. Some studies suggest that promoting and interpreting inductive reasoning in the classroom is a complex task for teachers. Herbert, Vale, Bragg, Loong, and Widjaja (2015) reported that elementary school teachers have little understanding of the distinctive aspects of the mathematical reasoning types and how to encourage mathematical reasoning in the classroom. Furthermore, noticing and interpreting the actions of students' reasoning in generalization tasks is complicated for both pre-service and in-service teachers (Callejo & Zapatera, 2017; El Mouhayar, 2018; Melhuish, Thanheiser, & Guyot, 2018). De Koning, Hamers, Sijtsma, and Vermeer (2002) claimed that elementary school teachers have difficulty focusing on the inductive process when teaching mathematical structures because attention is paid to the content or to students' responses and not to the process itself.

Rott and Leuders (2016) reported that teachers have the epistemological belief that inductive reasoning justifies discovery in mathematics over deductive reasoning at a 2:1 ratio. Recently, Rott (2021) showed that an inductive belief prevails over a deductive belief in more than half a group of secondary school teachers, but those teachers did not provide arguments for their belief. According to this author, it is necessary to investigate the consequences of such epistemological beliefs in the mathematics teaching. To provide information in this direction, the aim in our study was to analyse secondary school teachers' perceptions about inductive reasoning associated with their interpretations of this reasoning in teaching the quadratic equation.

Negative, or inadequate, perceptions of teachers concerning mathematics could unfavourably affect students' learning (Rosli et al., 2020). Therefore, our study contributes to identify whether the teachers' perceptions about inductive reasoning are adequate or not to encourage this type of reasoning in their students. It is desirable that teachers have clarity about inductive reasoning phases that go along with the transition from the particular to the general for discovering properties, knowledge, and general rules in mathematics.

In this regard, some authors have pointed out the phases and inductive processes people use to generalize from particular cases. Polya (1967) proposed four phases: observation of particular cases, conjecture formulation, generalization, and conjecture verification. Cañadas and Castro (2007) developed an empirical model of secondary school students' inductive reasoning that expands the phases referred to by Polya and comprises the following seven phases: working with particular cases, organisation of particular cases, search for and prediction of patterns, conjecture formulation, generalization, and demonstration. Sosa, Aparicio and Cabañas (2019) reported that mathematics teachers managed to generalize inductively when they connected three cognitive processes: observation of regularities, the establishment of a pattern, and generalization formulation.

We assume that if teachers have inadequate perceptions or little understanding of inductive reasoning, they will have difficulty in promoting this reasoning in teaching. Besides, there is a gap in the literature concerning the secondary school teachers' perceptions about inductive reasoning, even when this type of reasoning is a means of mathematical learning and it is possible to develop it starting in elementary school (Molnár, 2011; Molnár, Greiff, & Csapó, 2013; Papageorgiou, 2009). These factors led us to ask: What are secondary school teachers' perceptions of inductive reasoning? And how do they interpret it in teaching the quadratic equation concept?

METHOD

This research is qualitative, exploratory, and interpretative. It is exploratory because perception and teaching of inductive reasoning of in-service teachers is a little-studied topic. There are only a few approaches to this topic from a cognitive perspective or from teachers' epistemological beliefs in the literature. An interpretative approach was considered to generate categories of teachers' perceptions and to identify ways in which teaching is carried out by interpreting and making sense of the

characteristics attributed to inductive reasoning by teachers in a written and an oral way (Freitas, Lerman, & Park, 2017). The collection of data on the perception and interpretation of the teachers was carried out with an open questionnaire and an interview, both written and oral.

Context and Participants

This study was conducted with the participation of secondary school in-service mathematics teachers from Mexico; they were invited to participate in a professional teacher development program in mathematics through an open call. The program aimed to develop the teachers' inductive reasoning and to encourage them to enact this kind of reasoning in learning activities. Before the program began, 22 teachers—14 women and eight men—were selected among the teachers enrolled in the program; they agreed to participate in the study. The criteria for their selection were: (i) to have at least one year of experience teaching patterns and quadratic equations; (ii) to have the mathematical knowledge to solve tasks of generalization of quadratic patterns by inductive reasoning, whether acquired during their professional training or in training courses for teachers; and (iii) to know about inductive reasoning and mathematical generalization. The data for the selection of the participants were obtained from the academic information given by the teachers on the registration sheet for the program.

These criteria are explained by the fact that mathematics teachers have difficulties in generalizing quadratic patterns, as is reported in the literature. Besides, inductive reasoning is one of the mathematical reasoning types necessary to solve quadratic pattern generalizing tasks (Cañadas, Castro, & Castro, 2009; Rivera & Becker, 2016). The quadratic equation concept was chosen because, in the mathematics curriculum in Mexico, it is associated with the activity of generalizing quadratic patterns (Ministry of Public Education, 2017). In relation to the mathematical standards of the NCTM (2014), the aim of the mathematical activity in secondary school in Mexico (grades 7–9, ages 12–14) is to develop abilities such as generalization; abstraction; and inductive, deductive, and analogical reasoning. The students are expected to learn how to model linear, quadratic situations and to define patterns through algebraic expressions (Ministry of Public Education, 2017).

Data Collection

The data were collected in two working sessions. In the first one, the teachers gathered in a classroom and were asked to answer a written, open questionnaire, individually and simultaneously. This questionnaire was used to collect the responses of the participants about their perceptions of inductive reasoning and how these perceptions were brought to teaching. The questionnaire had two items, A and B, as shown in Figure 1. To obtain information about inductive reasoning perceptions, item A asked the teachers to write at least two characteristics of the reasoning in mathematics. Item B was oriented toward increasing understanding of how teachers interpret inductive reasoning to teach a mathematical concept. Therefore, item B asked participants to describe the phases to be followed to teach some aspect of the quadratic equation concept in an inductive way. The responses given by the

teachers to this item were expected to be based on an interpretation of the characteristics mentioned in item A.

A. State what, in your opinion, would be two or more characteristics of inductive reasoning in mathematics.

1.	
2.	
3.	

B. Based on the characteristics provided, indicate how phases 1,2, etc. would be in teaching and learning the concept of quadratic equation focused on inductive reasoning. Provide an example of each if possible.

Phase 1 Phase 2 Concept of quadratic equation

Descriptions of phases:

Phase 1:

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Phase 2:

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Figure 1. Questionnaire for data collection

Unlike closed or multiple-choice questionnaires, which contained predetermined responses given by the researcher that can skew the thinking of the study subjects, an open questionnaire contributes to a broader and more genuine picture of the perception of the participants (Ashton & Roberts, 2006; Peterson, 2000; Zohrabi, 2013). Thus, the teachers were asked to answer the written questionnaire to give them a greater opportunity to express themselves freely and to correct or complete their answers. There was no time limit for answering the questionnaire. This questionnaire was an adaptation of a previous questionnaire administered to a group of teachers with characteristics similar to those of the participants in this study to explore whether they knew the content (inductive reasoning and quadratic equation) of items A and B, and whether they understood what was requested.

The participants were called for an interview in the second session. During the interview, one of the researchers (first author of this paper) posed questions in an individual and ordered manner to the participants about some words, phrases, or sentences that the teachers used in their responses to the questionnaire. The purpose of this interview was to expand, clarify, or verify their written information and to avoid ambiguities or inadequate interpretations of the written responses on the part of the researchers. The audio of the answers during these interviews was recorded and transcribed for the researchers' analysis, together with the data obtained from the questionnaire.

Data Analysis

A thematic analysis was conducted to describe the teachers' perceptions based on the written and oral answers to item A. The result of this analysis was the generation of categories of teachers' perceptions about inductive reasoning. Then, the responses given to item B were associated with these categories and contrasted with the conceptual framework to identify how teachers interpret inductive reasoning in teaching the quadratic equation concept.

The thematic analysis method consists of identifying, analysing, organising, and systematically obtaining patterns (themes) in a data set by detecting and making sense of the experiences and meanings shared in a group (Braun & Clarke, 2006; 2012). This method was used to identify patterns of meanings in the common characteristics that teachers attribute to inductive reasoning and to form categories related to their perceptions. The six phases of the method were as follows: 1) familiarising yourself with the data, 2) generating initial codes, 3) searching for themes, 4) reviewing potential themes, 5) defining and naming themes, and 6) producing the report (Braun & Clarke, 2012, pp. 60–69).

Phase 1 of the analysis consisted of repeatedly reading the written answers to item A and repeatedly listening to the audio with oral responses. This phase helped in developing an initial overview and making notes about the teachers' ideas concerning inductive reasoning. In phase 2, codes were assigned to extracts of written responses and audio transcripts with key phrases or with characteristics of the inductive reasoning mentioned by the teachers; nine codes were obtained (Figure 2). It should be clarified that a teacher's response could refer to different perceptions of reasoning. The response included more than one code in these cases, and therefore, the number of codified excerpts was larger than the number of participants. In phases 2, 3, and 4, MAXQDA (2018.2) software was used to encode data, group the excerpts of responses by codes to look for themes, and search for the ones with potential.

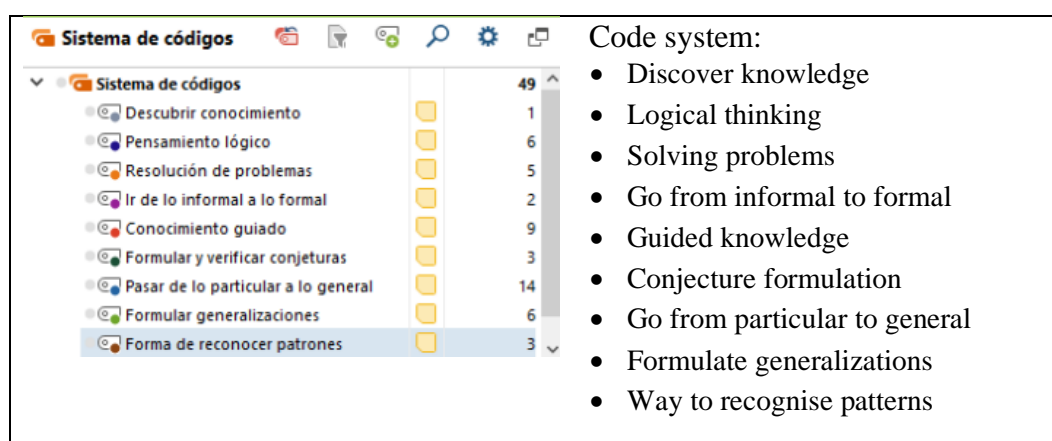


Figure 2. List of codes generated in MAXQDA (2018.2)

During phase 3, the generated codes and the excerpts associated with them were grouped and reviewed to search for themes that represent possible categories of the perception of the inductive

reasoning of the teachers. For example, the codes 'way to recognise patterns,' 'formulate generalizations,' and 'formulate and verify conjectures' were grouped to form a category that refers to generalizations' formulation and verification.

In phase 4, the themes were recursively reviewed in the context of the codes and total set of responses. The members of the research team became involved in the review and exchange of information during the codification process, searching for themes and defining categories. The main author of this work carried out the first part of these processes in each phase. Another researcher reviewed the generated information later, and finally, the team came together to define the codes, themes, and final categories. In this way, during phase 5, the five categories concerning the teachers' perceptions about inductive reasoning were defined and named. Categories were defined by selecting excerpts of the responses to analyse, clarify, and exemplify each category and to name the resulting categories. Finally, a report for this paper (phase 6) was generated.

RESULTS AND DISCUSSION

The thematic analysis resulted in the detection of five categories of secondary school teachers' perceptions about inductive reasoning. These categories represent patterns of shared meanings among the participants, according to the characteristics that they attributed to this type of reasoning. The following sections present the title, a brief description, and some excerpts of representative responses for each participant perception.

Categories of Perception about Inductive Reasoning

Category A: Way to Acquire Mathematical Knowledge

Teachers perceive inductive reasoning as a pedagogical method of leading students to achieve new knowledge. For them, inductive reasoning consists of posing a problem and, based on the students' previous knowledge, asking key questions so that students acquire new knowledge, similar to the Guided Discovery learning (Honomichl & Chen, 2012). The following excerpts are examples of this perception:

Teacher C: *It involves the use of previous knowledge so that it can be applied in a more complex situation or to generate new knowledge.*

Teacher L: *Give students an exercise and, based on their previous knowledge, allow them to draw their own knowledge. Have students brainstorm to learn what they know.*

Teacher M: *One of the characteristics is to begin asking key questions for the exercises and introducing students to the topic. Students begin to reason about the topic through questions and can visualise the previous knowledge. Guide questions. During the class, doubts may emerge (...) and questions may be asked to reinforce the student's reasoning (...), students can achieve the appropriation of concepts, processes, etc.*

Teacher V: *Students can come to a conclusion or definition based on their ideas or previous knowledge.*

These were the teachers' predominant perception, though they differ from the function of inductive reasoning as a teaching pathway for the formation of mathematical concepts. The difference is that the teachers did not perceive the function of reasoning as recognising the particular characteristics or attributes of the concept from a set of situations and encapsulating it in a general attribute (Davydov, 1990; Klauer, 1996; Sosa, Cabañas, & Aparicio, 2019); instead, they described issues of the guided discovery so the students could organise and generate their own knowledge through interrogation and group discussion (Yurniwati & Hanum, 2017). This category shows that teachers do not perceive the relationship between the underlying cognitive processes of inductive reasoning and mathematical procedures as something central to the acquisition of new knowledge.

Category B: Cognitive Process

In this category, the teachers perceived reasoning as a process that allows moving from particular instances (e.g. ideas, particular cases, or specific situations) to infer a general conclusion or result. More than half the teachers revealed an adequate perception of inductive reasoning as a cognitive process that involves inferring laws or general rules through observation of particular instances (Haverty et al., 2000). The following excerpts show this perception:

Teacher B: *Start from particular cases to get to general cases. Other cases that meet the observed characteristics are obtained. Conjectures about the observed cases are formulated.*

Teacher E: *It goes from the particular to the general.*

Teacher N: *It is a type of reasoning that consists of moving from particular to general ideas. Starting from concrete ideas to ideas in general. Generalize based on experiences of the given results.*

This perception, very common among teachers, concerns an inherent characteristic of inductive reasoning: It goes from the particular to the general (Reid & Knipping, 2010). Teachers perceive the starting and ending point of inductive reasoning; generalization is recognised as an intrinsic element for this type of reasoning. However, they little or nothing allude to the specificity of the processes that allow continuous progress from the particular to the general; only a few participants described inductive phases or processes such as observing regularities, establishing patterns, and the formulation of generalizations (Polya, 1967; Sosa, Aparicio, & Cabañas, 2019).

Category C: Generalizations Formulation and Verification

Almost a quarter of the teachers associated inductive reasoning with the formulation of generalizations and referred to the experimental character of this reasoning to verify the produced generalizations (Polya, 1967; Soler-Álvarez & Manrique, 2014). That is, they perceive that inductive reasoning is associated with the ability to predict the overall behaviour of specific cases and verify their truthfulness.

Teacher responses referred to how to obtain a generalization and verify it, as can be seen in the following excerpts:

Teacher A: *[In inductive reasoning] students analyse certain characteristics that are repeated continually under specific conditions. It is that they achieve generalizations, establish some rule or generalization based on what is repetitive, verify the established statements (...) I believe that you can establish a statement, and it could be wrong, after the verification; if you said that it was continually happening, for example, for positive numbers, and something else happens for negative numbers(...), you must to prove that it is always repeating; but if you find a case that does not go in the same way, then the generalization will not work. It is like testing if this is real, if this is true.*

Teacher B: *After seeing specific and concrete cases, you can try to predict what is coming next—for example, in a sequence, make conjectures and try to prove them. Predict those conjectures, see if they can be proved, and finally, come to a generalization.*

Bills and Rowland (1999) argued that inductive reasoning is a means for producing mathematical generalizations from particular cases. Thus, Category C differs from Category B, in the sense that reasoning is characterised in terms of generalization as a product of the process of inductive reasoning (Klauer, 1990). According to Fernández-León, Gavilán-Izquierdo, and Toscano (2021), in-service teachers are used to informal reasoning (based on examples) in the justification and generalization (or conjecture) processes. This could be the case because the teachers have a slightly superficial perception of this reasoning type as a means for mathematical generalization. Therefore, inductive reasoning is only perceived globally as a means for predicting and proof in mathematics, but the punctual aspects of this reasoning are not considered. Broadening this perception could help teachers with the development and identification of mathematical conjectures based on empirical data (Cañadas, Deulofeu, Figueiras, Reid, & Yevdokimov, 2007).

Category D: Strategy for Solving Problems

In this category, teachers associate inductive reasoning with problem solving and perceive it as a strategy for obtaining and arguing for the solution. Some of the excerpts where this perception was identified follow.

Teacher P: *They are the premises that allow us to conclude the resolution of problems. It is the form of reasoning that allows us to argue the resolution of problems by induction.*

Teacher H: *Each student should try to solve the posed problem with his previous knowledge.*

Teacher O: *Considering hypotheses or some propositions as starting points to solve a problem.*

Teacher Q: *Establishing a process of resolution based on several cases or examples.*

Inductive reasoning is a useful strategy for solving mathematical problems of categorization, number series, similitude between objects and relationships, and generalization, among other problems (e.g., Csapó, 1997; Molnár, 2011; Tomic, 1995). Furthermore, it facilitates recognition of similitudes in the structure of mathematical problems and generalization of methods of resolution when students work with situations that have different contexts but the same underlying structure (Sriraman & Adrian, 2004). Nevertheless, as in the study of Herbert et al. (2015), the teachers seem to be less aware of the relationship between mathematical reasoning—inductive, in our case—and problem resolution than they were in the previous categories. In particular, we find that these teachers omitted the description of the inductive strategy for solving a mathematical problem; neither pointed out the potential of this reasoning for recognising methods of solving problems that have the same structure. As a consequence, this limited perception might not be enough to promote problem solving skills in students.

Category E: Logical Thinking

Some teachers perceived inductive reasoning as a part of logical thinking—that is to say, as a way of reasoning based on rules and the performance of orderly and coherent procedures. They mentioned the following relevant characteristics:

Teacher J: *It emerges as part of a logical thinking process.*

Teacher R: *Reasoning must be logical—I mean, in an orderly and coherent way. It must follow certain rules to carry out the exercises.*

Teacher S: *It [inductive reasoning] is that the students develop logical thinking, that they understand what they do and perform the procedures in order.*

Category E suggests that the teachers must have a broader perception of inductive reasoning in logic such that they identify and establish inferences based on particular premises and recognise the probable character of the obtained conclusions or propositions (Hayes, Heit, & Swendsen, 2010; Reid & Knipping, 2010). This perception is associated with the fact that the teachers envision inductive reasoning as insufficient for validating mathematical propositions and believe that deductive proofs are needed (e.g. Conner, Singletary, Smith, Wagner, & Francisco, 2014; Martinez & Pedemonte, 2014).

The five categories of perception of inductive reasoning reveal that it is perceived in a very general way as a means and as an instrument to guide the acquisition of new knowledge and to make generalizations. However, the importance of the elements that constitute this form of mathematical reasoning is overlooked, specifically, the observation of regularities, the recognition of patterns, and the formulation of a generalization. Thus, while most of the teachers perceive generalization as a process and product of inductive reasoning, very few show clarities in this sense.

Interpretation of Inductive Reasoning in Teaching

The teachers' description of the phases for teaching quadratic equations led to the identification

of four different interpretations of inductive reasoning in teaching. Two of these interpretations are associated with the perception of inductive reasoning as a way to acquire knowledge (Category A) and as a cognitive process to move from the particular to the general (Category B). The other two observed that the ways of teaching in the responses of the teachers are not inductive in nature; one of these forms belongs to deductive reasoning, and the other one was named iconic. [Table 1](#) shows each interpretation and the number of teachers that expressed each interpretation.

Table 1. Inductive Reasoning Interpretations in the Teaching of Quadratic Equations

Interpretation	Frequency	Teachers
Way to acquire knowledge	8	C, E, F, K, L, M, S, T
Inductive (from the particular to the general)	4	B, I, N, R
Deductive (from the general to the particular)	6	A, D, O, P, U, V
Iconic	4	G, H, J, Q

Eight teachers' interpretation was that teaching a concept focused on inductive reasoning consists of guiding students to move from an existing or informal knowledge to a new knowledge, mainly through questioning or examples. This was the case for teacher M explained in [Table 2](#).

Table 2. Phases for Teaching Quadratic Equation Proposed by Teacher M

Phase	Description
1	<i>Previous knowledge: Introductory questions about algebraic expression, algebraic language, power, the law of exponents.</i>
2	<i>Application of the concept of 'basic' shape areas (with square shapes).</i>
3	<i>Delete data and replace it with literals. Start with formulas.</i>

The phases proposed by teacher M are coherent with her perception of inductive reasoning as a way to acquire knowledge. She verbally emphasised the importance of starting with the previous knowledge of the students and using questions to guide them to the definition and expression of a quadratic equation:

Recover their previous knowledge; tell them that they had already worked with linear equations, but that there are other types of equations. After, I propose a daily life situation that leads students to represent a square; then I'm going to ask questions that guide them to the relationship of the figure with the formula of the area and make them pose the equation. Tell them that it is the quadratic equation.

Certainly, inductive reasoning is a way to generalize knowledge by making inferences about unknown cases and new situations based on existing knowledge (Hayes, Heit, & Swendsen, 2010), but

the way teachers consider incorporating it into teaching is not appropriate. Even when the teachers interpreted inductive reasoning as a way of teaching to acquire new knowledge, the phases proposed for teaching quadratic equations did not include inductive actions designed to recognise the structure of the equation in different situations or contexts that could allow for the identification of an essential quality of the concept (Davydov, 1990; Klauer, 1996), such as the quadratic behaviour of the variables.

A minority of participants (only four teachers) described the teaching phases in line with inductive reasoning. These phases involve actions concerning the observation of particular situations; the search for and recognition of invariant characteristics of the situations; and a generalization based on a formula, equation, or definition (Cañadas & Castro, 2007; Polya, 1967). For example, teacher B proposed four phases (Table 3) associated with the phases mentioned by Polya (1967). The phases indicated a way to move from the particular to the general, even when the teacher did not specifically refer to the quadratic equation or give examples to illustrate the phases.

Table 3. Teaching Phases based on Inductive Reasoning Described by Teacher B

Phase	Description
1	<i>Specific cases or situations that can be quantified, manipulated, or visualised are provided.</i>
2	<i>Different cases that meet the observed characteristic or property are proposed.</i>
3	<i>It is required to predict that this characteristic or property will be fulfilled for other cases that are not tangible or directly observable.</i>
4	<i>A rule or formula that covers all possible cases is obtained—that is, a generalization.</i>

Although the teachers perceived the transition from the particular to the general as a feature of this reasoning, the responses reveal a lack of clarity about the underlying processes. In this way, we consider that these teachers, like the elementary school teachers (De Koning et al., 2002), need instruction about questions or tasks that allow them to shift from the content per se and focus on the processes to develop inductive reasoning in the classroom.

On the other hand, six teachers evidenced confusion about teaching based on inductive reasoning, since the order of the proposed phases refers to deductive reasoning (from the general to the particular). That is, the first phase presents a definition, characteristic, or general formula of the quadratic equation, and the other phases lead to something particular, be it an example or the solution of a specific quadratic equation. Table 4 shows the responses of teacher V to illustrate this deductive interpretation in teaching quadratic equations.

Table 4. Teaching Phases according to Deductive Reasoning Described by Two Teachers

Phase	Teacher V
1	<i>The characteristics of the quadratic equations are shown to the student.</i>
2	<i>Some examples are then shown; students will have to associate them with the quadratic equation of the form $ax^2 + bx + c = 0$</i>
3	<i>The student will have to find the unknown in the equation through factorisation.</i>
4	<p>a) <i>First, the equation is written $ax^2 + bx + c = 0$. Example: $3x^2 + 2x + 8 = 0$</i></p> <p>b) <i>The expression is then factorised in linear factors. $(3x - 4)(x + 2)$</i></p> <p>c) <i>Set each factor to zero: $3x - 4 = 0$, $x + 2 = 0$</i></p> <p>d) <i>Find the value of x: $x = \frac{4}{3}$, $x = -2$</i></p>

Although teachers have a theoretical knowledge of inductive reasoning, it is insufficient to enable them to carry it out in their teaching practice; they are more familiar with teaching using a deductive approach than an inductive one. Even when pre-service and in-service teachers tend to believe that the discovery of mathematics knowledge is inductive (Rott, 2021), the common teaching sequences of some teachers are still in line with deductive reasoning.

The teaching phases proposed by four teachers did not differentiate between inductive and deductive reasoning; instead, they offered an iconic type of treatment. Table 5 shows the phases proposed by teacher G as an example of this type of teaching.

Table 5. Teaching Phases for Quadratic Equations Proposed by Teacher G

Phase	Description
1	<i>Starting with the area of a square, the student must use his previous knowledge. Area (A) equals side by side (l); area equals the square of the side. $A = l \cdot l$ $A = l^2$</i>
2	<i>For example, given the figure of a square, what is the length of the side of the square if its area equals 400 square metres? And if the area is 100 square metres?</i>
3	<i>Make the figure bigger, adding different measurements to the sides of the squares so that they form rectangles or a bigger square.</i>
4	<i>Then write x instead of the measurement of the side of the square, so its area equals x^2, a squared number.</i>

In these cases, the phases involved the representation of a quadratic equation by a square or rectangular figure and, sometimes, a squared number. The teachers could have used this geometrical approach and inductive reasoning to obtain a general property: Every quadratic equation may be expressed as the product of linear factors of its roots; but the teachers focused on associating the degree

of the equation with the area of squares and rectangles.

The types of interpretation of inductive reasoning in the teaching of the quadratic equation are consistent with the categories of perception. Although the secondary school teachers perceived positive qualities of inductive reasoning in teaching and in the mathematical thinking of the students, most of the teachers in the group showed confusion or an inadequate interpretation of this reasoning when describing the teaching phases. Consequently, most of the teachers' interpretations are inadequate to foment the acquisition of the quadratic equation concept through this reasoning as they relegate the associated phases.

This could be the case because the teachers ignore the principles that guide the development of mathematical reasoning in the students based on generalizations and justifications in classroom (Mata-Pereira & da Ponte, 2017). In addition, the data suggest that the teachers are not aware of the processes (search for attributes and relationships, comparison of similitudes and differences among attributes, resolution and control) involved in the connection of knowledge in an inductive way (De Koning & Hamers, 1999; De Koning et al., 2002; Klauer, 1996).

CONCLUSION

This research identified five perceptions about inductive reasoning among secondary school teachers, which reflects that little clarity and sensibility are present regarding this type of reasoning in the teaching of mathematics. Although these perceptions are positive, teachers need to enhance their interpretations of inductive reasoning if they are to develop such reasoning in the classroom. Results suggest that it is necessary to confront and broaden secondary school teachers' knowledge about inductive reasoning to develop their teaching competency. In particular, it would be important for teacher learning and professional development programs to help clarify the use of this reasoning in the mathematical concepts' formation, along with recognising and articulating inductive processes in contexts of mathematical generalization and problem solving. In efforts to enhance understanding of the use of inductive reasoning in teaching, the results of this study could be used to investigate the relationship between the resolution and the use of tasks involving inductive reasoning by secondary teachers and the type of perceptions those teachers have.

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