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EXPLORING PROSPECTIVE ELEMENTARY MATHEMATICS TEACHERS' KNOWLEDGE: A FOCUS ON FUNCTIONAL THINKING

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Abstract

The importance of students being acquainted with algebraic ideas before secondary education has been revealed in the research literature. It is therefore essential that prospective elementary teachers (PTs) be prepared to instill an early algebra perspective in their teaching. However, PTs often show difficulties in algebra content knowledge, which need to be diagnosed aiming to assist them in developing the required knowledge to teach according to that perspective. This study aims to understand what aspects of functional thinking Spanish and Portuguese elementary PTs exhibit at the beginning of their teacher education program. The findings show that although PTs from both countries use different strategies to generalize functional relationships, the occurrence of successful strategies is low. Also, most participants provide local approaches in their interpretation of relationships between variables and reveal difficulties in understanding and connecting different representations of functions. These difficulties show that PTs lack important knowledge about functional thinking. By providing a framework concerning the functional thinking required for PTs to teach within an early algebra perspective, we shed light on a necessary step for teacher education programs to diagnose PTs' functional thinking and to assist them in developing the needed mathematical knowledge to teach accordingly.

Keywords: Early Algebra, Functional Thinking, Generalization, Prospective Teachers' Knowledge

Abstrak

Pentingnya pengenalan ide-ide aljabar siswa sebelum pendidikan menengah telah terungkap dalam penelitian literatur ini. Oleh karena itu, calon guru sekolah dasar (CG) harus siap untuk menanamkan perspektif aljabar awal dalam pengajaran mereka. Namun, CG sering kali menunjukkan kesulitan dalam pengetahuan konten aljabar, yang perlu didiagnosis dengan tujuan membantu mereka dalam mengembangkan pengetahuan yang diperlukan untuk mengajar sesuai dengan perspektif itu. Penelitian ini bertujuan untuk memahami aspek-aspek berpikir fungsional apa yang diperlihatkan oleh, CG Spanyol dan Portugis di awal program pendidikan guru mereka. Temuan menunjukkan bahwa meskipun CG dari kedua negara menggunakan strategi yang berbeda untuk menggeneralisasi hubungan fungsional, kejadian strategi yang berhasil rendah. Selain itu, sebagian besar peserta memberikan pendekatan lokal dalam interpretasi mereka tentang hubungan antar variabel dan mengungkapkan kesulitan dalam memahami dan menghubungkan representasi fungsi yang berbeda. Kesulitan ini menunjukkan bahwa CG kurang memiliki pengetahuan penting tentang berpikir fungsional. Dengan memberikan kerangka kerja mengenai pemikiran fungsional yang diperlukan untuk CG untuk mengajar dalam perspektif aljabar awal, kami menjelaskan langkah yang diperlukan untuk program pendidikan guru untuk mendiagnosis pemikiran fungsional PT dan untuk membantu mereka dalam mengembangkan pengetahuan matematika yang dibutuhkan untuk mengajar sesuai.

Kata kunci: Aljabar Awal, Pemikiran Fungsional, Generalisasi, Pengetahuan Calon Guru

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In face of well-known problems with the introduction of algebra, usually at ages of 12-13 years-old, there is a growing awareness of the need for engaging students with algebraic ideas earlier, by offering them opportunities to explore intuitive and informal ways of analyzing relationships between quantities, noticing structure in patterns, and studying change, in line with an early algebra perspective (Carraher & Schliemann, 2019; Stephens et al., 2017). In fact, early algebra is becoming part of the mathematics curriculum of elementary grades in different countries (Kieran et al., 2016), establishing the important learning goal of developing students' algebraic thinking as a capacity of making and expressing generalizations (Kaput, 2008). In that perspective we may include the notion of functional thinking, as it represents a form of generalization that involves exploring relationships between quantities that vary together (Blanton & Kaput, 2011).

Teaching in elementary school according to that perspective may represent a great challenge for prospective teachers (PTs) as most of them did not have that kind of experience as students and, therefore, they are not familiar with the algebraic ideas they are required to convey in their future practice (Magiera et al., 2013; McAuliffe & Vermeulen, 2018). However, there is still scarce research illuminating the Specialized Content Knowledge (SCK) (Hill et al., 2008) that elementary PTs effectively need to develop in their preparation to foster students' algebraic thinking, particularly concerning aspects of functional thinking, as well as how teacher education programs may address these issues (Hohensee, 2017; Rodrigues et al., 2019). Moreover, to better document the PTs' functional thinking and difficulties, Lannin et al. (2006) emphasize the need to be attentive to the commonly used frameworks to classify students' approaches to pattern generalization and to adapt them for characterizing PTs' knowledge regarding functional thinking.

This recommendation is particularly important in countries like Portugal and Spain where the elementary mathematics curriculum emphasizes ideas with some resonance with an early algebra perspective, particularly regarding functional thinking, but where there is not a consolidated practice around that perspective in schools (Morales et al., 2018; Oliveira & Mestre, 2014). Carrying out research in two national contexts, with their specificities, may contribute to better evaluate the suitability of a common framework on functional thinking for elementary PTs. Hence, this study aims to understand what aspects of functional thinking Spanish and Portuguese elementary PTs exhibit at the beginning of their undergraduate preparation, when solving algebraic tasks involving functional thinking. Specifically, this study addresses the two following questions: 1) How do PTs generalize functional relationships in patterns? and 2) How do PTs interpret variables and the relationships between them in different representations?

By providing a framework concerning the functional thinking required for PTs to teach within an early algebra perspective this paper intends to shed light on a necessary step for teacher education programs to diagnose PTs' functional thinking and to assist them in developing the needed mathematical knowledge to teach according to that perspective.

Functional Thinking as a Strand of Early Algebra

The important role of functional thinking as a gateway into early algebra, has been emphasized in the context of teaching interventions with elementary students (Carraher & Schliemann, 2019; Stephens et al., 2017). It does not imply a "formal" approach to functions in the sense of introducing and defining function as an object, rather it is a process that involves generalizing relationships between co-varying quantities and expressing those relationships in different representations as well as to use these to interpret and predict function behavior (Blanton & Kaput, 2011; Stephens et al., 2017). Thus, students will be able to build, describe, and reason with and about functions (Blanton & Kaput, 2011).

Generalisation is at the core of functional thinking and has been widely studied in the context of students' exploration of patterns, namely sequences tasks (Kieran et al., 2016; Radford, 2008). This process has received increasing attention in elementary mathematics syllabuses, through the introduction of numerical sequences, presented in different representations such as pictorial or geometric patterns or in numerical tables, as content to be taught, as these describe functional relationships that may support later the study of functions (Apsari et al., 2019; Blanton et al., 2011). In the context of algebraic patterns, the focus of generalization involves ideas such as: (i) grasping a commonality in some cases, (ii) expanding this commonality to all terms; and (iii) discerning a rule or schema to directly obtain any term of the sequence (Radford, 2008).

The literature has evidenced different generalizations strategies when students work with growing sequences, some of them expressing some difficulties, namely applying recursive reasoning that does not allow one to find a general rule or applying incorrectly a whole-object strategy (Moss & McNab, 2011). However, students may regard the difference between consecutive terms not simply in a recursive way, allowing them to determine a direct expression for the sequence. That happens when one uses the common difference as a multiplying factor and makes an adjustment of the result, in the case of non-linear sequences (Barbosa & Vale, 2015). However, sometimes this is introduced to students as a rule without fostering its understanding (Wilkie, 2016).

An important support to students' generalization in sequences is their presentation as a spatial configuration (commonly labeled as 'geometric pattern'). In these situations, students may link the spatial and numerical structures of the sequence, relying upon a visual approach, to realize the commonality in the cases and generalize it to all terms (Radford, 2011).

Conceptualizing and Representing Functional Relationships

In the context of early algebra, two main ways of conceptualizing functional relationships have been considered: covariational thinking and correspondence relationship (Blanton & Kaput, 2011; Confrey & Smith, 1994). According to Thompson and Carlson (2017, p. 423), covariational reasoning means "reasoning about values of two or more quantities varying simultaneously" and has been present in the mathematicians' way of thinking conducting to the modern function definition, although it has not been considered an explicit mathematical concept. Covariational approach may be associated with the notion of coordination of movement between values in the range which means: "being able to move operationally from y_m to y_{m+1} coordinating with movement from x_m to x_{m+1}" (Confrey & Smith, 1994, p. 137). Therefore, in covariational thinking one analyses how two quantities vary simultaneously and thus change becomes a central part of the description of the function (Blanton & Kaput, 2011; Ellis, 2011; Kieran et al., 2016). In the case of numerical sequences, it may lead to the multiple of difference generalization strategy, especially when it is represented in a table.

The correspondence approach to function, the most common in school mathematics, derives from the modern definition of function (Thompson & Carlson, 2017). In opposition with the covariational approach, the correspondence is a static one. Some of the difficulties recognized in meaningful interpretation of functions in opposition to the memorized rules and procedures, that often characterize students' work, may be a consequence of the dominance of this approach. Nevertheless, with the appropriate supportive instruction, even elementary students can describe correspondence in terms of functional rules (Oliveira & Mestre, 2014; Stephens et al., 2017) and thus there are recognized affordances in both covariation and correspondence approaches.

Central to a functional thinking perspective, is how students express the relationships between quantities, represent the associated generalization, and reason with multiples representations such as words, tables, diagrams, graphs, or symbols (Blanton et al., 2011; Kieran et al., 2016; Kusumaningsih et al., 2018), using conventional alphanumeric expressions or idiosyncratic symbols. Several studies show that children can make use of symbolic notation, with understanding, for expressing generalization of functional relationships that are presented in different forms (Blanton et al., 2011; Oliveira & Mestre, 2014). However, an overemphasis on a static view of functions may limit students' ability to generalize those relationships, such as being able to build an equation from a graph depicting a contextual situation by considering how the two involved quantities change simultaneously (Ellis, 2011). Thus, students need to interpret the graphs by local processes, that is focusing point-by-point, but also in a global way, by identifying a trend (Leinhardt et al., 1990). Another difficulty, when students explore reality situations, is that they often perceive the graphical representation as a picture of the physical situation, therefore, they should be given opportunities to explore these different representations so that "graphs become not just visual configurations, but structures embedded with meaning about relationship" (Blanton & Kaput, 2011, p. 16).

Pre-Service Teachers' Knowledge of Functional Thinking

Students' mathematical learning is commonly impacted by teachers' knowledge. Therefore, issues of teachers' content (subject matter) knowledge need to be uncovered and considered both for in-service and pre-service mathematics teacher education (Hill et al., 2008; McAuliffe & Vermeulen, 2018), illuminating how to support PTs to achieve the knowledge they ought to have for developing their future students' functional thinking. In the teachers' knowledge model of Hill et al. (2008), the SCK is seen as a type of content knowledge that enables teachers to "accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures and examine and understand unusual solution methods to problems" (p. 378). Although there are several studies focusing on students' algebraic thinking, the research is still scarce on PTs' algebraic thinking abilities, especially their strategies, misconceptions and difficulties related to the diversity of aspects of functional thinking (Yemen-Karpuzcu et al., 2017).

A few studies targeted PTs' knowledge regarding some algebraic topics related to generalization (including its formulation, representation, and justification), interpretation and use of algebraic symbology, and understanding of functions. Some findings, summarized by Strand and Mills (2014), show that although PTs are often able to generalize numerical and geometric patterns, they tend to have difficulties in interpreting and using efficiently the algebraic symbols. In their strategies to develop algebraic general rules in tasks involving linear, exponential, and quadratic situations, elementary PTs started by drawing and counting to support their thinking and used mainly chunking and recursive strategies (Alajmi, 2016). The challenges that PTs face in generalising explicit rules using symbolic algebraic notation is also frequently documented (Zazkis & Liljedahl, 2002). Even though PTs may express generality using algebraic symbolism, they often struggle in providing justifications for their reasoning what may express a memorization of procedures (Kieboom et al., 2014; Richardson et al., 2009).

Concerning functions understanding, the studies summarized by Strand and Mills (2014) also show that elementary PTs perform well on procedures related to linear functions. However, they have difficulties in interpreting the graphical representations, particularly when one variable is speed, since they confuse it with distance, and in translating between representations (symbolic and visual) and between a representation and its framing context. Still, difficulties in defining a variable and interpreting what it represents, are also documented in the research carried out with PTs (Brown & Bergman, 2013). Moreover, identifying the relationships contained in algebraic expressions and distinguishing between unknowns and variables constitute two aspects of functional thinking that are conceptually challenging to PTs (Hohensee, 2017).

The above studies adopted different frameworks to analyse PTs' knowledge and difficulties concerning specific aspects of functional thinking, such as generalization strategies, the use of algebraic notation to express this generality, and the interpretation of variables and graphical representations. However, as also emphasized by Lannin et al. (2006), to better document or evaluate PTs' proficiency concerning functional thinking it is necessary to develop frameworks covering broader core aspects involved in this process. In this study, to investigate PTs' functional thinking regarding the strategies to generalize functional relationships, the interpretation of variables and relations between them, and the connections between representations, we developed a framework, described in the next section, that encompasses and articulates different aspects that have been discussed above.

METHOD

Participants

The participants in this study are 94 (35 male and 59 female) Spanish and 70 (2 male and 68 female) Portuguese PTs who attended the 1st year of a degree in elementary education teaching, respectively at a public university in North of Spain and at two public high schools of education (labeled by A and B) in the center of Portugal. These institutions were chosen as a purposeful sample among those where we knew PTs had not received any specific teaching on algebraic thinking and teacher educators agreed to collaborate. All the PTs volunteered to participate in the study and had little or no exposure to early algebra previously in their elementary education.

Instrument and Data Collection

A questionnaire to assess PTs' algebraic thinking was developed by the researchers (authors). The six tasks that integrate the questionnaire were selected from the literature and adapted by modifying their statements and including new items. Then, the questionnaire was evaluated by eight specialists and trialed with pilot samples of elementary PTs from both countries (not participants in the study). These outcomes were discussed by the authors and further adaptations to the questionnaire were made to refine the structure and wording of the items in the tasks. The questionnaire was then applied in both countries at the beginning of the school year. In this paper, we focus on data collected from participant PTs' answers to three tasks from the questionnaire that involve functional thinking.

The "Geometric pattern" task (Figure 1), used to answer the research question 1, asks for a generalization of a geometric growing pattern (linear), allowing PTs to use diverse strategies to establish relationships between quantities and to describe and represent those relationships using multiple representations.

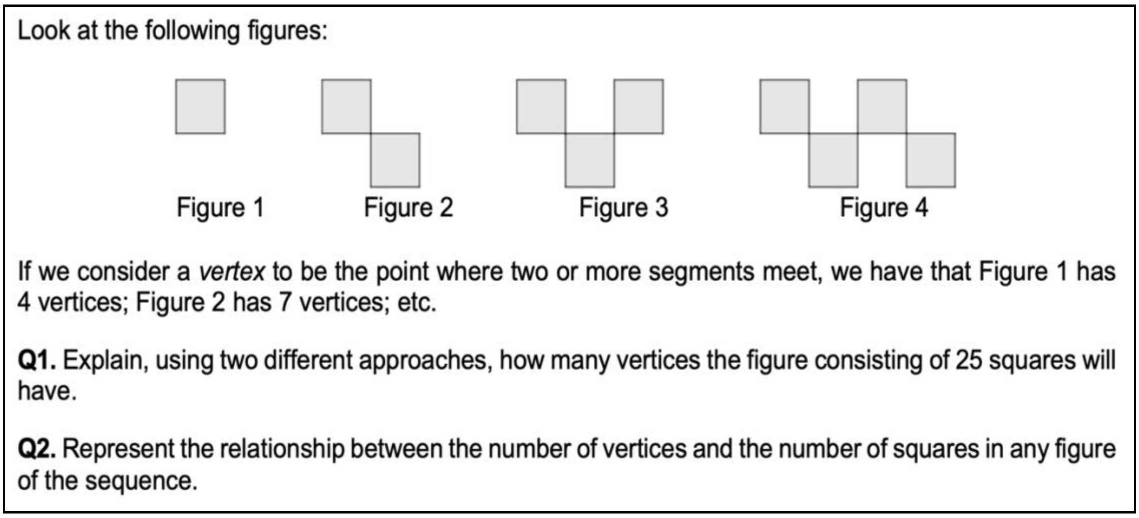


Figure 1. Geometric Pattern Task (adapted from Blanton et al., 2011)

The "Representations" task (Figure 2) intended to assess aspects of PTs' functional thinking such as: interpretation of variables and of relations between variables using co-variation and correspondence approaches to generalization; and connections between representations to interpret relationships.

Here are three solutions to the following problem: "Find the real values *a* and *b* that hold *a*+2=*b*+10". Indicate which solution (or solutions) you consider correct, and which incorrect. Justify your answer.

Solution 1:

"The solution are the numbers in the table"

a	b
8	0
9	1
10	2
11	3
12	4
13	5

Solution 2:

"The solution is a=10 and b=2 because 10+2=2+10"

Solution 3:

"The solution are the points on the line r"

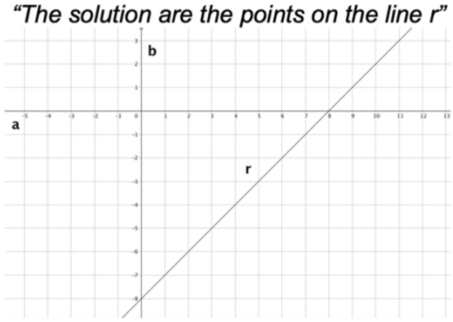
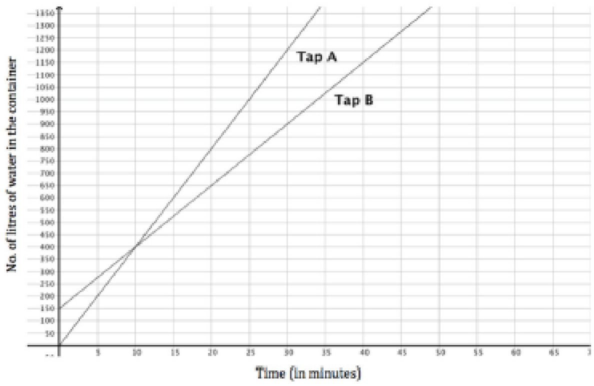


Figure 2. Representations Task (adapted from Hart, 1981)

Similarly, "Deposits" task (Figure 3) concerns the PTs' interpretation of variables and connections of different representations to interpret functional relationships, using covariation or correspondence approaches that could be local or global. So, both tasks were used to answer the research question 2.

To fill two water containers (container A and container B), both with a capacity of 1000 liters, two taps were opened (tap A and tap B respectively), with constant flow. The following graph shows the number of litres in each container over time (in minutes) since the opening of the taps. Provide a justified answer to the following questions:



Q1. Describe what happens with the containers at time 0.

Q2. What does the intersection point of the two straight lines of the graph relative to taps A and B represent?

Q3. What is the deposit that reaches its maximum capacity more quickly? After how long since the opening of the tap?

Figure 3. Deposits Task (adapted from Branco, 2013)

Data Analysis

This qualitative study followed a descriptive and interpretative analysis (Erickson, 1986) of PTs' solutions of the tasks, including a description of quantitative data organized in frequency tables. The pre-established categories we considered to interpret the PTs' functional thinking and the difficulties emerging from their work (Table 1 and Table 2) come from prior research on students' functional thinking, as described in the previous sections, particularly those from Barbosa and Vale (2015) and Leinhardt et al. (1990), with adaptations in the language used according to Ayalon et al. (2016). We include for each category a description of possible approaches which are task-specific, to generate a full picture of PTs' answers. Altogether the categories concerning generalizing functional relations and interpretation of variables and relationships provide a framework to characterize PTs' knowledge regarding functional thinking within an early algebra perspective.

To identify possible strategies to generalize a functional relation (both for distant generalization and a general term) in "Geometric pattern" task, we considered four categories (Table 1).

Table 1. Categories for Coding Strategies to Generalize Functional Relations

Category	Description: The PTs…
Counting	draw the next figures and count their elements
Difference	
Recursive	continue the sequence using the numerical difference between consecutive terms or explicit the recursive relation between consecutive terms.
Multiple of difference	use the difference between consecutives terms as a multiplicative factor (adjusting or not the result) to obtain distant terms or the general term.
Multiplicative reasoning	
Missing value	use the rule of 3 to find a distant term.
Proportional	use multiplicative strategies, starting from one known term of the sequence to find distant terms or the general term.
Correspondence	
Visual	express a relation between the two varying quantities for a distant term or in the general term, based on the characteristics of the pictorial representation.
Numerical	express a relation between the two varying quantities for a distant term or in the general term, based on the numerical sequence.

The categories used in "Representations" and in "Deposits" tasks (Table 2), attempt to capture the ways in which PTs interpret variables and relationships between them, including the connections between representations they establish. Interpretation in these categories means the action by which a PT gains meaning from a graph, equation, or context (Leinhardt et al., 1990).

The PTs' answers were independently coded by the authors, focusing on the identification of the categories proposed. To assure the validity of the analysis and to increase the reliability of the results, a check-scoring of PTs' answers of a random selection of 10 Spanish and 10 Portuguese initially coded questionnaires was undertaken by the authors to reach consensus. An inter-rater reliability was calculated for this sample of data covering all questions of the tasks and in all the codes an agreement of at least 82% was found among authors, which is considered satisfactory. Divergent interpretations or doubts concerning a codification were discussed until full agreement was reached.

Table 2. Categories for Coding Interpretation of Variables and Relationships

Category	Description: The PTs…
Interpretation of variables	interpret the variable as a *varying quantity* or as *unknown* in an equation.
Interpretation of relationships between variables	
Covariation	coordinate the two variables mentioning how dependent and independent variables change simultaneously rather than mentioning them separately, connecting different representations to identify this relation.
Global Correspondence	identify and/or explain the direct relation between two variables, connecting different representations to identify this relation, focusing on patterns, and gaining meaning about the relationship between variables.
Local Correspondence	identify and/or explain the direct relation between two variables, connecting different representations to identify this relation, determining when specific events or conditions are met.
Connection between representations	connect two representations to interpret a third one.

In the next section, we present in tables a quantitative descriptive analysis of the PTs' answers to each of the tasks and illustrate our interpretation of the ideas associated to each category with detailed examples of their answers (mentioned as S# in case of Spanish PTs and PA# or PB# for Portuguese PTs to assure their anonymity). The examples may provide evidence of PTs' functional thinking or the difficulties they reveal in their work.

RESULTS AND DISCUSSION

Geometric Pattern Task

The incidence of the different strategies that were used by the PTs from both countries are presented in Table 3. In some cases, PTs provided two different approaches in each of the questions (the distant term and the general term of the sequence), which were all considered in the analysis. For the distant term question, a total of 88 different answers (A) for Portuguese PTs and 125 for the Spanish were registered, as they were asked to present two different strategies. Regarding the general term, we found 36 answers for the Portuguese PTs and 69 for the Spanish ones. Next, we discuss these results separately for the distant term and the general term.

Table 3. Strategies Used by PTs for Finding the Distant Term and a General Term

Category		Distant term		General Term	
		Portuguese (A= 88)	Spanish (A=125)	Portuguese (A= 36)	Spanish (A=69)
Counting		15 (6)	14 (3)	–	–
Difference	Recursive	33 (25)	11 (2)	47 (0)	35 (0)
	Multiple of difference	19 (15)	6 (6)	25 (22)	7 (7)
Multiplicative reasoning	Missing value	8 (0)	20 (0)	0 (0)	7 (0)
	Proportional	1 (0)	1 (0)	8 (0)	0 (0)
Correspondence	Visual	18 (16)	40 (30)	11 (8)	35 (26)
	Numerical	1 (1)	2 (2)	3 (0)	1 (1)
Uncategorized		5 (0)	6 (0)	6 (0)	15 (0)
No answer (% of the PTs)		17	9	50	29

Note. The first number on each cell gives the percentage for each strategy, and between brackets the percentage of correct answers.

PTs' Strategies for Finding the Distant Term

Portuguese and Spanish PTs were able to provide a correct answer to find the distant term in 63% and 43% of the cases, respectively. The correspondence strategy was the preferred strategy to find the distant term for the Spanish PTs, which is present in 42% of their responses (32% correct), mostly through a visual approach. An example of a correct use of such strategy is the following PT's answer that expresses a relation between the two varying quantities for a distant term as referring to the characteristics of the pictorial representation: "2 squares lose one vertex, 3 squares lose two vertices, 4 squares lose three vertices, 25 squares lose 24 vertices. 25x4=100, 100-24=76" (S6). We also find incorrect implementations of this strategy, where the PT misses the extra vertex from the first square: "It will have 75 vertices since when you put 2 squares together, they share a vertex, so 25 squares x 3 vertices equals 75 vertices in total" (S10).

The difference-recursive strategy was most frequent among the Portuguese PTs (33% of their answers), leading them to a correct answer in most cases (25% of the total), as the example in Figure 4. We can observe that the PT begins by considering the number of vertices of the figure formed by 5 squares (16 vertices) and keeps adding 3 until reaching the figure with 25 squares, concluding: "The figure formed by 25 squares will have 76 vertices".

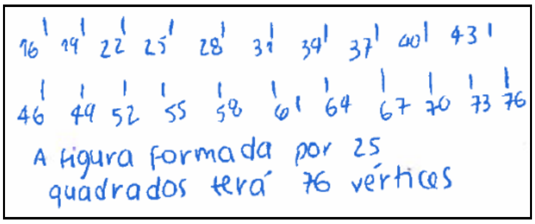


Figure 4. Example of Difference-Recursive Strategy - PB24

The difference strategy of type multiple of difference was the second most used (19% of responses) by Portuguese PTs to find the distant term. After realizing that there is a constant difference between the terms, these PTs used it as a multiplicative factor to determine the required distant term, making an adjustment at the end by adding the extra vertex of the first square. The example in Figure 5 shows that the PT realizes that the 25th term is obtained by multiplying 24 by 3 and then adjusting the result by adding up four from the first term. It is also worth noting that most Portuguese PTs who used this strategy obtained the general term already in their answer to this question, and applied it to correctly find the distant term, as the example in Figure 5.

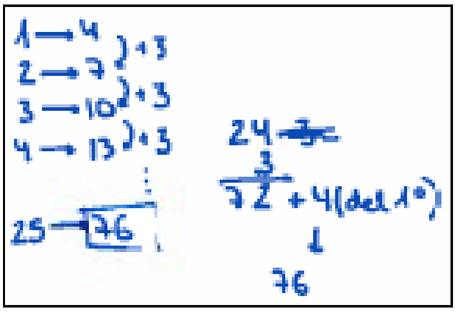


Figure 5. Example of Multiple of Difference Strategy - PA19

Multiplicative reasoning strategy of type missing value was used in 20% of Spanish PTs' answers to find the distant term. This strategy occurs when the participant uses the rule of three to find a distant term, which does not result in a correct solution since this is not a proportional numerical relation. For instance, one PT states: "By a rule of three: 1 square – 4 vertices; 25 squares – x. [So] x=100 vertices" (Figure 6). This strategy was not frequent among Portuguese PTs.

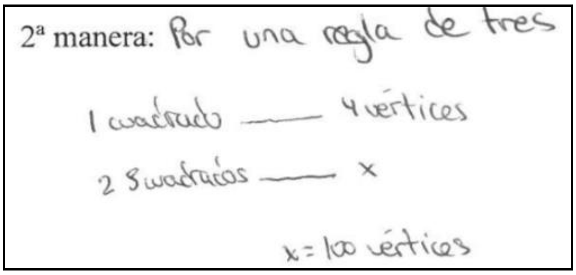
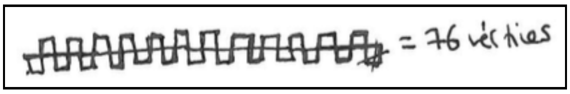
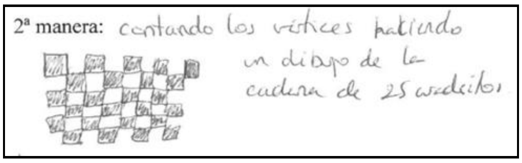


Figure 6. Example of Incorrect Missing Value Strategy - S8

The counting strategy based directly on the figural pattern, it is still present in about 15% of the answers, by both Portuguese and Spanish PTs. In most cases, this strategy led them to incorrect answers as PTs were not able to understand how the pattern grows. Two examples of Spanish PTs' answers, one correct (Figure 7(a)) and one incorrect (Figure 7(b)), are given below. For example, S77 (Figure 7(b)) explains: "By counting the vertices of a drawing of the 25-squares chain". Finally, we should remark that 17% of Portuguese PTs and 9% of the Spanish PTs did not provide an answer to the distant term question.



(a)



(b)

Figure 7. Examples of Counting Strategies - S7(a) and S77(b)

PTs' Strategies for Finding the General Term

Many PTs from both countries were not able to provide the general term: 50% of Portuguese and 29% of Spanish PTs did not answer this question. Only 30% of the Portuguese PTs and 34% of the Spanish who answered this question provided a correct general term for the sequence. For those cases, a description on how they represent the relationship (using words, syncopated language, or mathematical symbols) is presented.

We find that most Portuguese PTs' strategies (47%) were difference-recursive type, not leading to the general term, as also frequent among Spanish PTs' answers (35% of the strategies), like the one of S33 (Figure 8), who explains: "Every time we add a square, one of its vertices is shared with the previous, adding 3 vertices per square instead of 4, 1 less than in the case of the first".

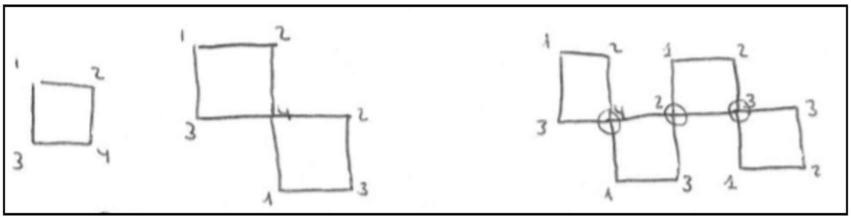


Figure 8. Example of a Difference-Recursive Strategy - S33

Portuguese PTs most successful strategy for the general term was the multiple of difference strategy (25% of the strategies), which led them to the correct answer in most cases (22% in total). As mentioned before, most of the PTs found a general term already when answering the question for the sequence's distant term. Thus, it seems that the PTs were using a procedure they have learnt to determine the general term of a sequence representing an affine relation between variables. In Figure 9, we can notice that the PT stresses the difference between consecutive terms ("+3"), and, using this difference as a multiplicative factor, writes down the general term in symbolic mathematical language. Then, the PT applies the relation to find the distant term.

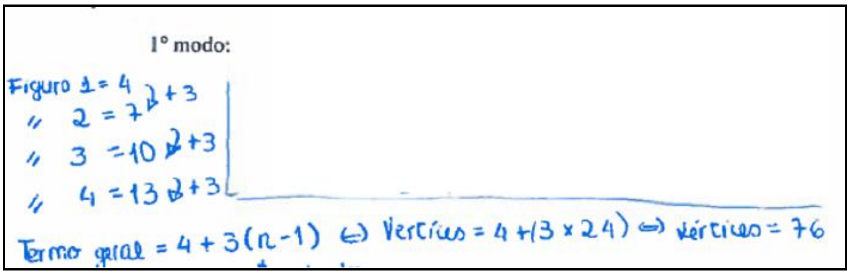


Figure 9. Example of Multiple of Difference Strategy - PB14

The Spanish PTs' most common strategy to find the general term was the correspondence strategy (35% of the strategies). These PTs used a visual approach in most cases, such as the following one: "Number of vertices [in each square] times the number of squares and then you subtract the number of squares minus 1" (S53). Here the PT expresses the direct relation between the two varying quantities in a general rule using words and referring to the contextual features of the sequence (squares and vertices). Therefore, the way the relationship between the variables is expressed, which could be written as $4\times n-(n-1)$, is dependent on how the PT has apprehended the structure of the spacial sequence.

It is also worth noting that very few PTs, among those who found the functional relationship, provided a full symbolic equation to express it, such as the one presented by S62 who wrote: "3n+1=v, n= nr squares, v=vertices".

Overall, these results showed that PTs from both countries encountered serious difficulties when trying to find the general term for the two varying quantities and adopted diverse approaches in the generalization of patterns: Portuguese PTs privileged the multiple of difference strategy, whereas the Spanish preferred a correspondence visual strategy.

Representations Task

The results concerning the approaches used by the PTs to interpret variables and relationships between them, including the connections between representations they established, were identified on the obtained PTs' answers, and are presented in Table 4. In the following sections, we discuss the three categories separately.

Table 4. PTs' Interpretation of Variables and Relationships between Variables and Connection between Representations

Category		Portuguese (N=70)	Spanish (N= 94)
Interpretation of variables	As a varying quantity	27 (11)	72 (11)
	As two specific unknowns	10 (0)	9 (0)
	No evidence of interpretation	20	10
Interpretation of relationships between variables	Global correspondence	13 (11)	28 (11)
	Local correspondence	21 (0)	44 (0)
	No evidence of interpretation	23	19
Connection between representations	Connect two representations to interpret a third one	33(11)	46 (11)
	No evidence of connection	24	45
No Answer (% of the PTs)		43	9

Note. The first number on each cell gives the percentage of answers for each category, and between brackets the percentage of correct ones. "No Answer" concerns the percentage of the PTs that did not answer to this task, thus in each category it completes the 100% of answers.

Interpretation of Variables

We can observe that 72% of Spanish PTs interpreted variables as a varying quantity (Table 4) but only 27% of Portuguese PTs showed such interpretation. An example evidencing this interpretation is the answer of a Spanish PT, who argues: "The correct solution is the third one, since for each value of a there is a value of b, with all possible options, including negative numbers, and the previous [solutions] are more limited since they only provide some solutions" (S33). This PT's answer shows an understanding that the two letters may assume many diverse values, as the solution offered by the graph, also providing a justification. Other PTs express the same interpretation of variables, referring to the diversity of values they may assume, but their responses were classified as incorrect since they consider the solution provided in the table as the correct one, as the following example of a Portuguese PT shows: "The correct solution is Solution 1 since it provides a sequence of possible options. Solution 2 is the incorrect solution since it only has one solution and there can be more" (PA13). These incorrect answers may result from the PTs' difficulties in interpreting the graphic representation and thus not considering it as a possible solution.

We also find that about 10% of both Portuguese and Spanish PTs interpreted the variables as two specific unknowns. This was demonstrated for instance by a PT who explains: "Solution 2 is correct because if a is 10, added to 2 gives 12, and if b is 2, added to 10 gives 12. They are symmetric. Solution 1 is wrong because there is only one situation in this table that meets a+2=b+10" (PA1). We interpret that the PT believes the relation between quantities to be adding up to 12, and that holds only for a specific value of each letter. Also, worth noticing is the incorrectness of the language used by the PT when stating that the numbers are "symmetric".

Finally, we observe that 20% of Portuguese PTs and 10% of the Spanish PTs showed no evidence of interpretation of the variables. In these cases, the PTs did not provide an explanation for their choice or seemed not to have understood the question.

Interpretation of Relationships between Variables

In this category, it was analysed how PTs interpreted the relationship between variables, distinguishing global correspondence approaches from local ones. We observed that 21% and 44% of the Portuguese and Spanish PTs, respectively, showed a local correspondence approach in their responses. This has been classified as incorrect as it does not consider that the relation between the values applies to an infinite set. We find an example, classified as incorrect, in the following answer: "The correct solution is solution 1 [table]. This is the one that includes all the values that make the expression a+2=b+10 true" (PA8). Here some of the values provided in the solutions are considered by the PT to hold the expression a+2=b+10, but there are again no indications to suggest that more than a finite set has been considered.

We find evidence of a global correspondence approach in 13% and 28% of the Portuguese and Spanish PTs, respectively. An example of a global approach is found in the following answer that refers to the infinity of the set of solutions: "The only one representing all solutions is solution 3, which defines well the R-representation of a line that has neither a beginning nor an end" (PA3). Some PTs also evidenced a global correspondence approach, but they did not consider the solution provided by the graph as the only one that is correct, as the case of a PT who only labels "Solution 1" as correct and explains that: "Any number a that exceeds in 8 any other number b corresponds to a+2=b+10 since 10 exceeds 2 by 8" (S2). They consider how both variables change in relationship to one another, and to the existence of many more values than those on the table, but not in the graph.

There are still 23% and 19% of the Portuguese and Spanish PTs, respectively, whose answers showed no evidence of interpretation. An example providing no evidence is the one by a PT who argues: [The first solution in the table] is not correct since zero cannot be in an equality" (S19), showing a lack of knowledge regarding algebraic expressions.

Connections between Representations

Only 11% of both Portuguese and Spanish PTs' answers showed, correctly, connection between representations. In these answers, the PTs understand the algebraic expression provided in the task and connected it with the table and the graph representations, emphasizing that the solution given by the graph is the only complete one, as showed in the following example: "I consider all solutions to be correct even though I consider solution 3 totally complete, since solution 1 and 2 provide examples of solutions, and solution 3 gives a more comprehensive solution" (PB7). Other answers, although showing connection between representations, were classified as incorrect since the PTs failed in not considering the graph as the only one that displays all solutions of the algebraic expression, like the following: "Solution 1 is correct since giving values to a or b, those numbers come out. Solution 3 is correct since it is a representation of the table of Solution 1" (S26).

The answers classified as no evidence of connection between representations (24% and 45% of Portuguese and Spanish PTs, respectively) mostly show a lack of understanding of the graphic representation. That is the case of a PT who explicitly states: "Solution 3 cannot be right since the right line r has nothing to do with a and b" (S47). Another example is given by a Spanish PT who argues about the graph in solution 3 as follows: "Incorrect: if you place the numbers of the line by the given equation, you obtain: a=8, b=-8; 8+2=-8+10, 10=2, which is not true" (S45), and another by a Portuguese PT who explains "This solution [solution 3] is not correct since -8+10=2 and 8+2=10. So, -8+10 ≠ 8+2" (PA2). It is noticeable here that the two PTs are using the same incorrect argument when rejecting Solution 3. They wrongly consider the point (-8,8) to be a point on the straight line and argue that the line cannot be a solution to the problem since that point does not hold the expression.

In summary, the results concerning the Representations task reflect many difficulties among PTs of both countries in all categories of the analysis. The Portuguese PTs had a lower level of participation than Spanish ones, in general, but a very low percentage of correct answers among PTs from both countries were observed. This brings to light important PTs' difficulties regarding the understanding of algebraic expressions and their representations.

Deposits Task

The results of the approaches used by the PTs to interpret relationships between variables given by a graph, identified on their answers to the three questions in this task, are presented in Table 5.

Table 5. PTs' Interpretation of Relationships between Variables

	Portuguese (N= 70)			Spanish (N=94)		
Interpretation of relationships	Q1	Q2	Q3	Q1	Q2	Q3
Co-variation	—	—	0 (0)	—	—	2 (1)
Correspondence	58 (56)	61 (49)	52 (26)	93 (81)	94 (83)	86 (40)
No evidence of interpretation	6	6	17	4	3	9
No Answer (% of the PTs)	36	33	31	3	3	3

Note. The first number on each cell gives the percentage of answers for each category, and between brackets the percentage of correct ones.

In the case of Q1, 56% and 81% of Portuguese and Spanish PTs, respectively, provided a correct answer (deposit A: empty, and deposit B: 150 liters). It was interpreted herewith that all PTs who were able to identify variables and their relationship used a local correspondence approach. Some answers were incorrect, like the following one: "Tap A is slower so at the instant 0 there are 0 liters, while Tap B comes out faster because at the same time there are already 150 liters" (S45). The PT can identify the coordinates, but he focuses on the taps' speed, which reflects difficulties in interpreting the variables. Other answers to Q1 were classified as no evidence of interpretation of relationships, like: "Tap A had nothing in its flow" (S22) or "Without water" (PA22). We take that the absence of explanations in these answers reflects a lack of understanding of relationships between variables.

Regarding Q2, we find that most PTs who provided an answer showed a correspondence approach (49% and 83% of the total of Portuguese and Spanish PTs respectively), like one Portuguese who argues "Within 10 minutes both deposits have 400 litres" (PB18), showing therewith to interpret the meaning of the variables and their relation from the graphic representation. About 12% of both the Spanish and Portuguese PTs showed correspondence approach in Q2 but their answers were considered incorrect since they established a relation between the time and the number of litres in the deposits but failed to explicitly refer to the variables, as observed in the following answer: "That at that time they had the same capacity" (S6). This PT uses the word "capacity" to mean 'fill level', a mistake quite frequent among Spanish PTs. It was also very common among PTs from both countries to allude erroneously to the taps in their justification instead to the deposits.

Few responses to Q2 were classified as providing no evidence of interpretation of relationships like in the following argument: "The moment in which both taps get the same flow and, from there, they have different rhythm" (S36). The PT shows not to understand the dependent variable (and therefore the relationship between the two variables) since he assumes that it represents the flow instead of the amount of water.

Both co-variation and correspondence approaches in the interpretation of relationships between variables were evaluated by Q3 where PTs were asked: (1) to name the deposit that reaches its capacity faster, and (2) the time when that occurred. Most PTs that provided an answer adopted a correspondence approach (52% and 86% of Portuguese and Spanish PTs, respectively). About half of these answers were correct, like the one by a Spanish PT who argues: "The deposit A is the one that reaches its maximum capacity 1000 litres more quickly, at 25 minutes, versus deposit B that reaches it 10 minutes later" (S38). The acknowledged correspondence between the variables is evident in some cases by the marks in the graph to connect the values of the coordinates of each point (Figure 10).

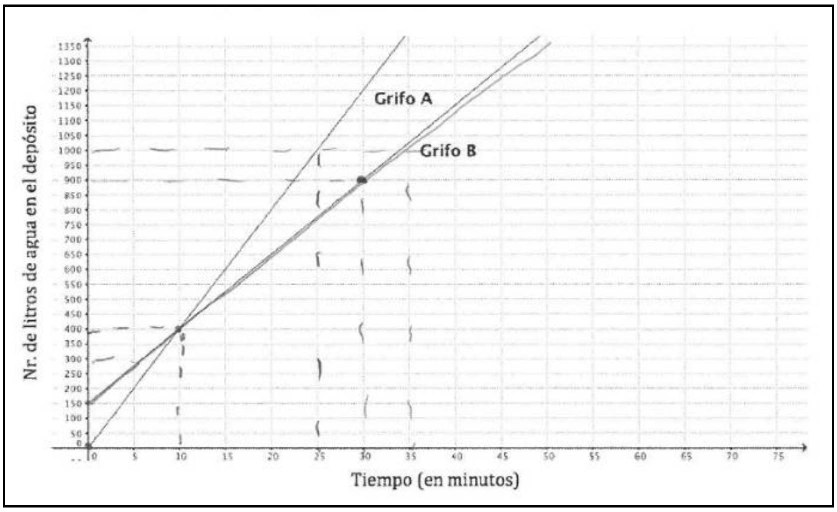


Figure 10. Example of Correspondence in the Graph - S38

Many of the mistakes occurred when the PTs considered the capacity of the deposits to be the top of the ordinate axis in the graph (1350 litres), and not the 1000 litres stated in the task, as shown in the answer: "A takes 35 minutes and B 50 minutes approximately" (S45). Just two Spanish PTs showed a co-variation approach. One of them argues that: "Tap A since it has a greater slope" (S17).

Few PTs provided incorrect answers when answering Q3 which were classified as no evidence of interpretation of relationship, such as this one: "Tap A has more power, so whatever its start, it fills first" (S22). It was common among Spanish PTs to make assumptions about contextual aspects that were not provided by the information in the graph, or by the task's statement.

This empirical study examined aspects of Spanish and Portuguese elementary PTs' functional thinking at the beginning of their undergraduate preparation, as they solved algebraic tasks. The results show that PTs from both countries have difficulties in generalizing algebraic rules and in interpreting variables and the relationship between them, as in other studies (e.g. Alajmi, 2016;

Hohensee, 2017). Nevertheless, when compared to previous research, this study gives a broader perspective about elementary PTs' functional thinking as it encompasses more core dimensions.

As elementary PTs need to develop a specific SCK for teaching in an early algebra perspective, being able to identify the relationship between the two variables in a sequence is of paramount importance. Addressing the first research question in this study, we found that among the Portuguese PTs there is a prevalence of recursive-difference strategies, both for determining the distant term and the general term of the sequence that may not comprehend a functional relationship. This approach may result from a memorization of a procedure, which is often identified in other studies (Kieboom et al., 2014; Wilkie, 2016). On contrary, the Spanish PTs who were able to determine the distant or the general term tend to use a correspondence approach, relying on the visual characteristics of the pattern, but the occurrence of successful strategies is still low among them. It is also worth noting that very few PTs from both countries used a full symbolic equation for the general term which may be an indicator of their difficult with the algebraic language, as stressed by Strand and Mills (2014).

In what concerns the second research question, specifically with a focus on the PTs' understanding of variables, we found that Spanish PTs considered variables as a varying quantity in a higher percentage than Portuguese PTs but had equally a low rate of correct responses. PTs from both countries provided local approaches in their interpretations of relationships between variables more often than global ones (Leinhardt et al., 1990). Particularly, Spanish PTs showed this approach very often by checking whether the values of the table fulfilled the relationship given by the algebraic expression. However, the tendency to see the table as a finite set of coordinated points, attached to a local view of function, may be linked to a static view of function, preventing PTs from searching how the values of the two variables change simultaneously (Ellis, 2011). In general PTs also reflect many difficulties in understanding the algebraic expression presented in the task's statement, and therefore hardly connect it with other representations, as also pointed out in other studies (Strand & Mills, 2014). In face of these two difficulties altogether it is not surprising that the majority of PTs from both countries have shown a lack of important knowledge regarding the graphical representation of a function, namely that it conveys all the range of coordinates for the function. When interpreting the relationship between variables that describe quantities from a real context represented in a graph, there is an increased number of PTs who can describe the functional relationship at a local level. Nevertheless, there is still an important number of Portuguese PTs who do not provide an answer and a majority from both countries who reveal difficulties in connecting the information about the variables provided by the graph with the real context. This finding suggests that using graphs to model concrete situations may not only entail great complexity for students (Patterson & McGraw, 2018), but also for many elementary PTs, and it may prevent them from promoting the exploration of such representations in their future teaching practice.

CONCLUSIONS

Although PTs can use different strategies to generalize functional relationships, the occurrence of successful strategies is low. Also, most participants provide local approaches in their interpretation of relationships between variables and reveal difficulties in understanding and connecting different representations of functions. These findings show that regardless of PTs' school experiences in mathematics and the differences among the curriculum in both countries, PTs lack important knowledge about functional thinking when they start their preparation to become elementary teachers. From this study, we can derive some implications for teacher education.

First, considering that elementary PTs may have quite diverse mathematical backgrounds, teacher educators need to understand the key mathematical ideas they have developed. A framework like the one proposed in this study may be a starting point to identify different aspects of PTs' functional thinking to understand their misconceptions and difficulties with algebraic ideas, as recommended by Yemen-Karpuzcu et al. (2017), as well as if they tend to rely on the use of rules and procedures, without understanding, when solving tasks that involve functional relationships. Second, exploring core ideas on functional thinking can also be used as a rich context for PTs to rebuild the mathematics they have learnt, namely by stablishing connection among topics they often see in isolation, as it happens sometimes with sequences and functions. An attention on PTs' functional thinking in an early algebra perspective is also essential for their understanding of the mathematics behind the tasks to propose to their future students.

Finally, and more specifically, opportunities should be given to PTs to: (i) reflect on the different strategies and their level of efficiency for generalization of functional relationships and (ii) deepen their understanding of functional relationships in different representations and how they connect with each other. In what concerns the first aspect (i), we may illustrate that the use of a difference-recursive strategy that has been mentioned in the literature has not led students to understand the structure of the patterns, nor the relationship between the quantities involved. However, using a multiple of difference approach can be a successful strategy when students understand the relation between the numerical sequence of values and the general term. At the same time, we want to support PTs' functional thinking, encouraging them to advance to more sophisticated strategies based on correspondence approaches, but still, as future teachers, they need to understand that simpler strategies are also important and can provide opportunities for students' further development. As most PTs did not experience this kind of activities as students, teacher education programs should provide opportunities for them to explore generalization in different contexts.

Concerning the second aspect (ii), teacher educators should create situations that allow PTs to understand how different representations may support thinking in functional terms, either in a covariational or correspondence approach. With the strong incidence in a static view of function, associated with the correspondence approach, throughout the middle and secondary schools (Ellis, 2011), elementary PTs may need to further explore tables and graphs as means to understand the relationship between the variables of a function, from a global perspective, embedded in significant contexts for elementary students.

Future research may address how the framework adopted in this study may support the design of teacher education programs for promoting PTs' functional thinking, regarded as an important dimension of SCK for elementary teachers, as well as other dimensions of PTs professional knowledge, namely the knowledge about their future students.

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