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FLAWS IN PROOF CONSTRUCTIONS OF POSTGRADUATE MATHEMATICS EDUCATION STUDENT TEACHERS

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Abstract

Intending to improve the teaching and learning of the notion of mathematical proof this study seeks to uncover the kinds of flaws in postgraduate mathematics education student teachers. Twenty-three student teachers responded to a proof task involving the concepts of transposition and multiplication of matrices. Analytic induction strategy that drew ideas from the literature on evaluating students' proof understanding and Yang and Lin's model of proof comprehension applied to informants' written responses to detect the kinds of flaws in postgraduates' proof attempts. The study revealed that the use of empirical verifications was dominant and in situations. Whereby participants attempted to argue using arbitrary mathematical objects, the cases considered did not represent the most general case. Flawed conceptualizations uncovered by this study can contribute to efforts directed towards fostering strong subject content command among school mathematics teachers.

Keywords: mathematical proof, transpose and multiplication of matrices, flawed conceptualisations, levels of proof comprehension

Abstrak

Berniat untuk meningkatkan pengajaran dan pembelajaran gagasan bukti matematika penelitian ini berusaha untuk mengungkap jenis-jenis kelemahan dalam pendidikan pascasarjana guru matematika siswa. Dua puluh tiga guru siswa menanggapi tugas pembuktian yang melibatkan konsep transposisi dan penggandaan matriks. Strategi induksi analitik yang menarik ide-ide dari literatur tentang mengevaluasi pemahaman bukti siswa dan model pemahaman bukti Yang dan Lin diterapkan pada tanggapan tertulis informan untuk mendeteksi jenis kesalahan dalam upaya bukti pascasarjana. Studi ini mengungkapkan bahwa penggunaan verifikasi empiris dominan dan dalam situasi. Dimana peserta berusaha untuk berdebat menggunakan objek matematika yang arbitrer, kasus-kasus yang dipertimbangkan tidak mewakili kasus yang paling umum. Konseptualisasi yang cacat yang ditemukan oleh penelitian ini dapat berkontribusi pada upaya yang diarahkan untuk menumbuhkan perintah konten pelajaran yang kuat di antara guru matematika sekolah.

Kata kunci: bukti matematis, transformasi dan perkalian matriks, konseptualisasi yang salah, tingkat pemahaman bukti

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Over the past decades research on mathematical proof has gained increased attention and most research studies have revealed that student teachers have a fragile understanding of mathematical proof (Bieda, 2010; Lesseig, 2016; Mejia-Ramos, & Inglis, 2009; Maya & Sumarmo, 2011; Noto, et al. 2019). However, a huge number of those studies were based on convincement issues. In other words, studies sought to determine how convincing a given argument would be to the participant (e.g., Bleiler, Thompson, & Krajčevski, 2014; Martin & Harel, 1989). Hence, there is scarcity of research that examines how students construct proofs of mathematical statements, particularly on students' competencies in resolving proof tasks.

Precisely, the argument is that little is known about students' reasoning as they immerse themselves with proof tasks as instructional and assessment strategies have tended to promote memorization and regurgitation of lecture notes (Ndemo, Zindi, & Mtetwa, 2017; Stylianou, Blanton, & Rotou, 2015). Hence, if we conceive proving as a problem solving process then arguments generated by students when they engage with proof tasks should illuminate the kinds of students' thoughts about mathematical proof (Lee & Smith, 2009). I stark contrast, many school and university students and even teachers of mathematics have only a superficial grasp of the idea of a mathematical argument (Jahnke, 2007; Brown & Stillman, 2009; Saleh, et al. 2018). Yet prospective secondary mathematics teachers need to exit teacher preparation with a firm grasp of the concepts of the concepts of transposition and multiplication of matrices (Brown & Stillman, 2009).

The notion of mathematical proof has persistently caused severe discomforts among teachers and learners at many scholastic levels. Hence, it is necessary that teachers engage in substantial learning of the concept of mathematical proof. Furthermore, mathematics education that encourages student teachers to engage in autonomous proof constructions is crucial for their learning in order to build their capacity to explain the concept in a persuasive manner to their future students (Jones, 1997).

To bring the research problem in proper perspective the researcher refers to Harel, Selden and Selden's (2006) comment about students' struggles with the notion of mathematical proof. Harel, et al. (2006) wrote:

> We know where the students are, we know where the mathematicians are, but we just don't know how to get mathematics students from where they are to where we want them to be (p. 148).

Harel et al.'s quote points to an undesirable gap between students' and experts' understandings of mathematical proof. Hence, one of the primary goals of mathematics instruction at tertiary level is to promote among student teachers conceptions of mathematical proof held by expert mathematicians. Therefore, mathematics education instruction should aim to enhance expert conceptions of the concept of mathematical proof among student teachers. In this regard, the researcher was impelled to explore student teachers' thinking as reflected in in-class mathematics problem solving tasks as student teachers engage with concepts deemed to be within their conceptual reach. The research problem is explained in the next section.

Statement of the Problem

In-service student teachers do not have a firm grasp and appreciation of the idea that proofs that explain can be more elucidating and help to foster justification. Doing proofs at school level and even at undergraduate level has been characterised by students and instructors resorting to rote memorisation and regurgitation of instructor notes. The researcher argues that if comprehension tests only ask students to regurgitate memorised facts then such students are likely to develop a superficial understanding of those mathematical facts (Mejia-Ramos, et al. 2012). Further, such students are likely to emphasise form over substance, that is, the ritual proof scheme becomes dominant (Harel & Sowder, 2007). Yet, teachers need a flexible and firm understanding of mathematics content they are supposed to teach (Shahrill, et al. 2018; Prahmana & Suwasti, 2014). The researcher reiterates that the notion of mathematical proof has been reported to develop reasoning, that is, the ability to think rationally and logically among learners. However, the ability to reason can be impeded by flaws in students' mental representations of the concept of mathematical proof (Garret, 2013).

Promoting argumentation skills can illumine the kinds of flaws in students' thoughts about the concepts of transposition and multiplication of matrices. It is in an argument that we likely to find the most significant way in which higher order thinking can manifest during mathematics learning (Jonassen & Kim, 2010; Putri & Zulkardi, 2018; Ahmad, et al. 2018). The goal of the current study is to gain insights into the kinds of limitations in postgraduate students' argumentation schemes–chunks of reasoning with respect to the notions of transpose and multiplication of matrices. Pertinent questions that therefore, come to mind in this respect are: how can we develop an understanding of the flaws in student teachers' mental representations of the idea of proof? What is the nature of these limitations among in-service mathematics teachers?

Generating answers to these questions can contribute to useful ideas for teacher education in Zimbabwe. While many studies on students' discomforts with the concept of a mathematical proof have been carried out most of such studies have been based on arguments participants find convincing from those availed by researchers (e.g., Bleiler, et al. 2014; Matin & Harel, 1989). Therefore there is scarcity of empirical studies based on students' own proof constructions, that is, their own actual voices which in turn could illuminate their thinking processes as they engage with proof and proving (Duval, 2006; Mejia-Ramos & Inglis, 2009; Ndemo, Zindi, & Mtetwa, 2017; Mumu, et al. 2018). This study intends to respond to this dearth in studies grounded in students' own proof constructions by addressing the research question, such as what kinds of flaws characterise postgraduate students' conceptions of the concepts of transposition and multiplication of matrices?

By addressing this question the study may provide mathematics teachers with a clearer picture of what is needed to help students to develop a good command of the concept of mathematical proof in order to teach it effectively to their future students. Furthermore, it was anticipated that evaluating postgraduate students' understandings of the notions of transposition and multiplication of matrices could in turn inform teachers and mathematics educators of what specific aspects they understand and what aspects they do not understand.

Furthermore, proof serves as a vehicle for discovering new mathematical ideas and if learners properly grasp the notion of proof then they learn from it (Mejia-Ramos, et al. 2012). Mathematical proof also serves the purpose of promoting reasoning skills (Weber & Mejia-Ramos, 2015), which in turn would contribute to the student's cognitive development about the concept of mathematical proof. In this regard, the study aims to inject new ideas into the growing body of theoretical frameworks and methodologies for understanding mathematical concepts.

Matrix Theory

If A is a mxn matrix of a field $K,$ of scalars then the transpose of matrix A is the nxm matrix whose rows are the columns of A in same order (Goodaire, 2014). The transpose of matrix A is denoted A^t. Hence, if $A = (a_{ij})$ then $A^t = (a_{ji})$ (Lipschutz, 1991). In other words, if

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \text{ then } A^t = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}.$$

First, we observe that the product AB of two matrices is somewhat complicated and hence there is need to describe prerequisite ideas for the definition of the product of matrices. The product $A \cdot B$ of a row matrix $A = (a_i)$ and a column matrix $B = (b_i)$ is defined as:

$A \cdot B = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{k=1}^{n} a_k b_k$. Second, we observe that the row matrix A and the column B should have the same number of elements for the product $A \cdot B$ to be defined. Finally, the product $A \cdot B$ is a scalar or a $1x1$ matrix (Lipschutz, 1991).

The researcher now describes the definition of a product of two matrices. Suppose $A = (a_{ij})$ and $B = (b_{ij})$ are matrices over the field of scalars $K,$ the product AB is defined if the number of columns of matrix A is equal to the number of rows of matrix B. If the number of columns of matrix A is equal to the number of rows of B we say the two matrices A and B are of compatible sizes (Goodaire, 2014). Thus, if matrices A and B are compatible say A is an mxp and B is a pxn matrix then the product AB is an mxn matrix whose ij-th entry is obtained by multiplying the i −th row A_i of the matrix A by the j −th column, B^j, of the matrix B. That is: $AB = \begin{pmatrix} A_1 \cdot B^1 & A_1 \cdot B^2 & \cdots & A_1 \cdot B^n \\ A_2 \cdot B^1 & A_2 \cdot B^2 & \cdots & A_2 \cdot B^n \\ \cdots & \cdots & \cdots & \cdots \\ A_m \cdot B^1 & A_2 \cdot B^2 & \cdots & A_m \cdot B^n \end{pmatrix}$. Alternatively, we can write the product as:

$AB = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{p1} & a_{p2} & \cdots & a_{pn} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{pmatrix}$, where $c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{ip} b_{pj}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. The researcher emphasizes that if A and B are not compatible then the product AB is not defined. So if A is an mxp matrix and B is a qxn matrix then the product AB exists only if $p = q$. This study sought to explore how postgraduate students could apply the notions of transpose and multiplication of matrices to determine whether the product $A^t A$ exists.

Yang and Lin's (2008) Model

The goal of the study was to uncover the sorts of flaws in postgraduate students' proof attempts. To assess in-service teachers' understandings of mathematical proof the researcher drew ideas from a model for assessing comprehension of mathematical proof by Yang and Lin (2008). Yang and Lin introduced what has come to be known as a model of reading comprehension of geometric proof (RCGP). The model consists of four levels which represent increasing levels of cognition. Next, the researcher briefly describes these levels.

First, there is the *surface level* of the RCGP model whereby a prover acquires basic knowledge regarding the meaning of the statements and symbols in the proof. For example, for the theorem*: A real sequence converges to a real number L if given* $\varepsilon > 0$, *the interval* $(L - \varepsilon, L + \varepsilon)$ *contains all but infinitely terms of the sequence* (a_n), surface level understanding of this theorem can include the conception of ε as a small radius, an understanding of a finite set and the basic idea that a sequence is a mapping with domain the set of natural numbers, (ℵ) and range in real numbers, ($\mathbb{R}$). The second level has been called *recognising elements*. At this level a prover should be capable of identifying the logical status of statements that are explicitly or implicitly involved in the proof construction exercise. The researcher now uses the example given at the surface level that concerns a theorem on the characterisation of converging sequences in $\mathbb{R}$ to describe proof understanding anticipated at this level. At this second level of Yang and Lin's YCGP model, a prover should recognise that for every $\varepsilon > 0$, there is a natural number, $N(\varepsilon)$, dependent on ε such that if $n > N(\varepsilon)$ then $|a_n - L| < \varepsilon$. A prover's chunk of reasoning at the second level should show also awareness that the interval $(L - \varepsilon, L + \varepsilon)$ contains infinitely many terms of the sequence (a_n).

At the third level there is what Yang and Lin refer to as *chaining of elements.* Central at this level is the fact a prover demonstrates his/her understanding of the way in which different statements are connected in the proof by identifying the logical relations between them (Mejia-Ramos et al., 2012). Following up on the example given to illustrate the YCGP model, the researcher describes the logical relations between statements a prover should depict in his/her argumentation. A prover should make the connection that since $\varepsilon > 0$ then there is a natural number, $N(\varepsilon)$, such that if $n > N(\varepsilon)$ then

$|a_n - L| < \varepsilon$

$\Rightarrow -\varepsilon < a_n - L < \varepsilon$

$\Rightarrow L - \varepsilon < a_n < L + \varepsilon$

$\Rightarrow a_n \epsilon\ (L - \varepsilon, L + \varepsilon)$ for $n > N(\varepsilon)$.

Hence, a prover who would have attained the third level of Yang and Lin's model of reading of comprehension of geometric proof (RCGP) should make a series of logical inferences illustrated. The third level of Yang and Lin's model is similar to Azarello's (2007) conceptualisation of proof as a sequence of inferences. Azarello (2007) in Mejia-Ramos and Weber (2014) view proof as a series of claims of the form $P_1 \to P_2 \to P_3 \to \cdots \to P_n$, where P_n is the consequent statement or conclusion while, $P_1, P_{2,} \ldots, P_{n-1}$ constitute the premises of the proof construction. In this conceptualisation of composing proofs the focus is on how each new inference is derived from the previous inference, for instance, how does claim P_3 lead to claim P_4 (Mejia-Ramos & Weber, 2014).

Finally, Yang and Lin's RCGP model has a fourth level called encapsulation whereby a prover is anticipated to interiorize a proof holistically and develop an appreciation and understanding of its application in other contexts (Mejia-Ramos & Weber, 2014). The researcher illustrates the encapsulation phase using the example of real sequence already given. A prover at the fourth level of proof understanding can draw ideas from this theorem and use them together with ideas from the completeness property of $\mathbb{R}$ to prove our first criterion for convergence that: *A bounded monotone sequence converges.* In their study, Yang and Lin focused on the first three levels of their model. The current study that involves postgraduate students seeks to explore those students' understanding of ideas drawn from Elementary Linear Algebra of Matrices. The concepts had been covered during their studies at undergraduate level. Hence, the study seeks to explore elements of the encapsulation level of Yang and Lin's model in postgraduates' solution attempts to the task.

Other Useful Ideas on Evaluating Students' Understanding of Mathematical Proof

Mejia-Ramos and Weber (2014) and Mamon-Downs and Downs (2013) concur that one way to evaluate whether a student understands a proof is by determining the student's proof behaviour or problem solving behaviour. In addition to scrutinising one's proof behaviour, Hanna and Barbeau (2008) suggest that a student's proof construction competence can be ascertained by determining the extent to which the student applies pertinent ideas to the proof task or theorem in other situations. It can be, therefore, noted that Hanna and Barbeau‘s suggestion on how to evaluate proof understanding is similar to the encapsulation level of proof conceptualisation proposed by Yang and Lin (2008).

Conradie and Frith (2000) stated that students often fail to grasp the meaning of terms when trying to comprehend a proof thereby hindering their ability to understand other aspects of the proof. To get a sense of students' grasp of key terms to a theorem, Conradie and Frith (2000) suggest that a researcher can ask students to define the key terms or sentences. This technique of determining understanding features at the surface level of Yang and Lin's (2008) model. Alternatively, students can be asked to identify examples that illustrate a given theorem or a term in a theorem. Generating examples that illustrate a theorem or ideas embedded in that theorem is similar to Yang and Lin's second level of their RCGP model. Finally, a prover should develop a firm grasp of the logical relationships of the statement being proven and the major assumptions and conclusions of the proof. In other words, proof understanding involves grasping the proof framework (Selden & Selden, 2009).

Morselli (2006) writes that generating examples has several benefits in proving. The benefits of generating examples to illustrate key concepts involved in constructing proof of a given theorem include illuminating the defining property of pertinent mathematical ideas to the proof being constructed, revealing logical connections in ideas embedded in the mathematical statement that can form the crux of the proof, and providing a pictorial representation of the mathematical ideas especially in proof situations where graphical instantiations can be found. However, despite these benefits, empirical verifications have a severe limitation that the statement can be true for particular examples a prover could have considered but can be false for just one instance not considered by the prover. This is so called notion of a counter example (Stylianides, 2011). Hence, empirical explorations do not provide complete and conclusive evidence about the truth-value of a statement a prover may seek to establish.

Finally, another useful idea in evaluating an individual's ability to compose proof relates to the structure of a mathematical argument. A mathematical argument consists of a connected sequence of assertions in which the consequent statement is called the conclusion and the rest of which is called the premises (Curd, 1992). The premises provide valid reasons for inferring that the conclusion is true. Furthermore, a mathematical proof is said to be valid if it is deductive, contains no errors and provides complete and conclusive evidence about the truth-value of a mathematical statement (Weber & Mejia-Ramos, 2015). The ideas presented in this section were important in evaluating students' responses to the task. For instance, arguments generated by mathematics education post graduate students were checked for logical consistency between premises and conclusions drawn.

METHOD

Research Design

A cross-sectional survey research design with a qualitative bent was used in this study (Flick, 2011). The intent of the study was to capture the in-service teachers' state of knowledge structures about the notions of multiplication of matrices and transpose of a matrix. Further, the study sought to investigate mathematical connections built by the postgraduate students which would in turn allow them to determine whether the product $A^t A$ was defined. Hence, the study was designed to get what Flick (2011) calls "a picture of the moment" (p. 67). Such picturing was based on the assumption that concepts examined were considered to be within the conceptual reach of the participants since they teach these at secondary school level and at teacher training colleges.

Study Informants

The study involved in-service mathematics education students who were studying towards a master degree in mathematics education. The cohort consisted of 23 members: 15 males and 8 females. Of the 15 male participants, 4 were lecturers from primary teacher training colleges while the rest were high school mathematics teachers with more than 10 years teaching experience. One out of 8 female in-service teachers was a lecturer at a primary teacher training college while the other 7 female student teachers were high school mathematics teachers with more than 8 years of teaching experience. Both college and high school mathematics curricula cover subject content on Elementary Linear Algebra of Matrices. Hence, the researcher had assumed on reasonable grounds that the concepts were within the conceptual reach of the participants.

The postgraduate programme in mathematics education offered by the university that served as the study site is now described. The master degree programme has duration of 2 years during which the in-service teachers major in mathematics content and professional courses. Professional courses deal with 3 mathematics education modules and a module covering foundations in science education.

In addition professional studies also include Research Methods and Statistics module that is designed to prepare postgraduate students for research projects during the second and final year of their studies. There are 5 modules under the Professional component of the postgraduate programme just described. Mathematics content courses include: Metric Spaces and Topology, Functional Analysis and Non-linear Differential Equations. Subject content courses drawn from the learning area of Statistics are: Multivariate Statistics, Survey Sampling Methods, and Operations Research. Mathematics content and Professional modules are covered during the first two semesters of year one.

During the third and fourth semesters, that is, during year two, student teachers engage in research projects. The content of the postgraduate programme articulated here supports this researcher's assumption that theory of matrix multiplication and the notion of transpose of a matrix were within students' conceptual reach. The study involved all the 23 students who had enrolled for the master degree studies. Data collection took place during week 10 of semester 2 of year one and this period was deemed strategic for data collection because lectures had been completed and students were doing individual studies and hence pressure had eased.

Research Instrument and Data Collection Procedure

A proof task with an open instruction: *If A is a mxn matrix, determine whether the product A^tA is defined. Justify your answer,* was posed. Although the task involved elementary concepts in Matrices, it was anticipated that the answer generated would not be a result of applying standard procedure involving a known mathematical result or a known theorem. In other words, an attempt was made to avoid assessing proven results in literature on Linear Algebra such as: (i) $(kA)^t = kA^t$ where k is a scalar (ii) $(AB)^t = B^tA^t$.

The argument here is that assessing students' competences at composing such proofs could possibly lead to regurgitation of proofs from textbooks and so would serve very little purpose with respect to the research goal of establishing the kinds of flaws in postgraduate mathematics education students' conceptualisations of mathematical proof. Instead of reproducing memorised theorems such as those stated here, the postgraduate students were expected to tap from their knowledge of matrix multiplication and the notion of a transpose of a matrix to decide whether the product A^tA is defined. It was also considered that these are well known mathematical ideas for the postgraduates. Hence, students could face no difficult in establishing the connections between these ideas. Thus, the task was intended to measure students' competence at autonomous proof writing.

Following Dahlberg and Housman (1997) task-based interviews were used to gather data. To collect data a task sheet with space for writing the answer was provided. Students responded to the task individually in the mathematics lecture room. No time restrictions were imposed. The participants took about 15 minutes working on the task. All task sheets distributed were returned with some text scribbled on by the students. During data collection process, participants were encouraged to document their thinking as much as possible and request for extra answer sheets when necessary.

Data Analysis

The data analysis procedure employed the analytic induction strategy (Punch, 2005). Analytic induction comprises a series of alternating inductive and deductive steps whereby data driven inductive codes are followed by deductive examination as described in the following steps. First, a marking guide was devised to evaluate postgraduate students' proof efforts. Second, the researcher then performed content analysis (Berg, 2009) of the students' written efforts. Content analysis was facilitated by a refinement of a classification originally developed by Stylianides and Stylianides (2009). The refinement was driven by the desire not only to identify correct proofs but to also explicate the different thinking styles displayed by the postgraduate student teachers. Following Stylianides and Stylianides' (2009) classification as well as a scrutiny of the students' written efforts, a data matrix with the following format was then constructed. Column 1 entries consist of categories identified from students' written responses, in column 2 each category is described, column 3 entries are frequencies of the categories. Steps 1 and 2 so far described led to a data matrix. The creation of the data matrix constituted the induction analysis part of the analytic induction strategy employed in this study.

Finally, the emerging categories from content analysis of postgraduates' proof attempts were mapped to levels of proof understanding in Yang and Lin's (2008) model and other ideas about proof understanding explained in Section 2.3 of this paper in order to ascertain postgraduate students' level of grasp of the concept of mathematical proof. The mapping of results of inductive content analysis constituted the deductive analysis part of the analytic induction strategy. Furthermore, in-vivo codes (Corbin & Strauss, 2008; Varghese, 2009) were used to support inferences made about postgraduate students' proof construction competences.

Ethical Considerations

Flick (2011) suggests that research should involve participants who have been informed about the aim of the study and that participation should be voluntary. Hence, with regards to informed consent the researcher explained to the postgraduate mathematics education students why research into the nature of students' flawed conceptualisations of mathematical proof was necessary as it could promote conceptual teaching of the mathematical ideas. The students were asked to complete informed consent forms. Further, the researcher emphasised that consent was to be given voluntarily (Flick, 2011).

Another ethical concern was about anonymization of data. Anonymizing data involved removing any identifiers from the students' responses (Punch, 2005). Furthermore, during research reporting pseudonyms were used to describe postgraduates' flaws in their conceptualisations of mathematical proof.

RESULT AND DISCUSSION

Inductive content analysis of postgraduate students' written responses revealed the categories summarised in Table 1.

Table 1. Emerging categories from content analysis of postgraduate students' proving attempts ($n = 23$)

Category	Description	Frequency
C1	Correct proof	4
C2	Empirical argument used	8
C3	Argument adduced does not represent the most general case	10
C4	Consequent statement missing or wrongly formulated	7
C5	Argument has algebraic slips	1
	Total	**30**

Table 1 shows that the dominant scheme of argumentation was one in which student teachers produced justifications that failed to offer complete and conclusive evidence about the fact that the product A^tA is defined for any matrix $A = (a_{ij})$ over a field of scalars K. From the same Table 1 it can be seen that postgraduate students' efforts were also dominated by use of specific examples, denoted by C2 with 8 out of 30 responses and a very significant number of responses (7 out of 30) were in the category in which the conclusion did not follow logically from the premises– represented by code C4. Finally, the same Table 1 illustrates that very few (4 out of 30) student teachers managed to justify the existence of the product A^tA – a worrisome result at postgraduate level, more so in light of the fact that elementary concepts of transpose and multiplication of matrices were explored. Presented next is a discussion of thinking styles displayed in each category summarised in Table 1 for the purpose of illuminating flaws detected in postgraduate mathematics education student teacher informants.

C1: Correct proof constructed

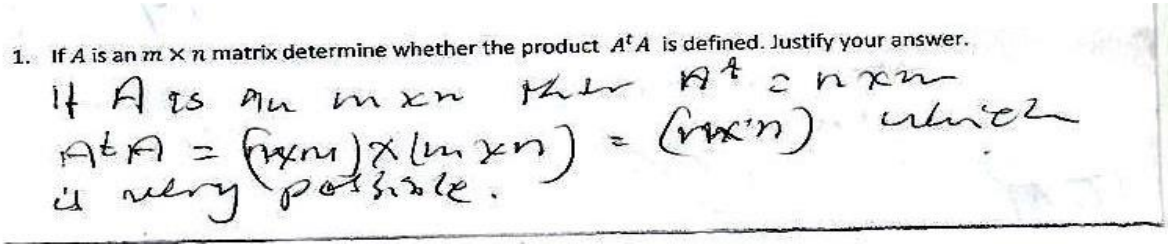

Figure 1: Trevor's written response to the task on multiplication of matrices

Figure 1 illustrates that Trevor could state the order of the transpose matrix A^t correctly as nxm. Trevor then tested for compatibility of the two matrices by determining whether the number of columns of A^t was equal to the number of rows of matrix A and then reached the conclusion that the product A^tA is defined. Hence, Trevor's written response shows that he had a firm of grasp of the concepts of transposition and multiplication of matrices. Further, there was proper chaining of these knowledge elements which then led to the valid conclusion that the product A^tA exists (Yang & Lin, 2008).

C2: Empirical arguments used

Typical examples in this category were produced by Munya and Tauya. Munya and Tauya's efforts are shown in Figures 2 and 3 respectively.

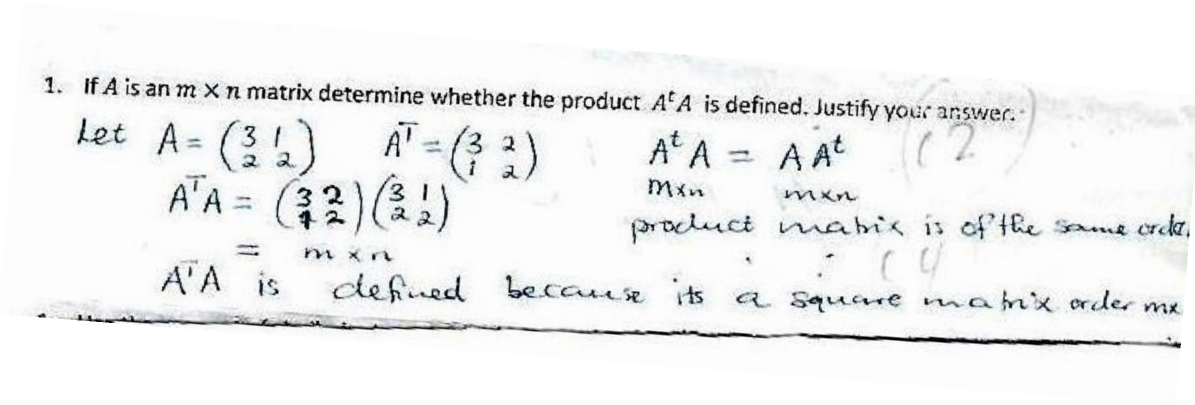

Figure 2. Munya' written response to the task

Figure 2 illustrates that Munya chose a specific example of a $2-$square matrix A and proceeded to write the transpose of matrix A correctly. The product A^tA is then equated to mxn, presumptively referring to the order. Munya went on to write $A^tA = AA^t$, which is a false assertion because a matrix A and its transpose are not commutative over the binary operation multiplication over the field K. It is a serious flaw in reasoning displayed by Munya. Further, Munya represented the order of the product A^tA to be mxn and the order of AA^t was written also as mxn. It was yet another flawed reasoning as Munya could not identify that the operation of transposing entails interchanging the rows and columns of a matrix.

Finally, Munya concluded that the "product matrix is of the same order." This is a flawed argument because for m by n matrix A the product A^tA is a $n-$square matrix while the product AA^t is a $m-$square matrix. The discussion of Munya's proof effort reveals that he had not even attained the second level of Yang and Lin's (2008) model of comprehension of geometric proof. Munya was operating at surface level of Yang and Lin's model as he could state the transpose of the $2-$square matrix he had written. Furthermore, Munya concluded on the basis of the specific example of the $2-$square matrix that "AA^t is defined because it a square matrix order m." The claim shows that Munya had not grasped the fundamental limitation that empirical verifications cannot be elevated to the status of a proof (Ndemo, Zindi, & Mtetwa, 2017; Stylianides, 2011). In other words, Munya exhibited a weak command of the notion of counter-argumentation in mathematics.

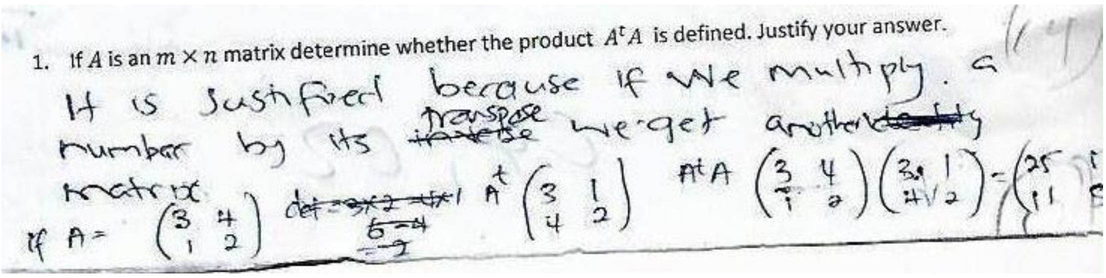

Figure 3. Tauya's written response to the task

From Tauya's solution attempt the word *number* was used to refer to a matrix when he wrote "when we multiply a number by its transpose we get another matrix." This claim by Tauya reveals lack of precision in the manner he used the word *number* which he used interchangeably with the word *matrix.* Further, Tauya's effort also reveals lack of deep grasp of the basic limitation that specific examples cannot be used to represent general matrix multiplication. In other words, although matrix multiplication held in the single instantiation considered, the specific example picked by Tauya should not be regarded as a proof. In terms of Yang and Lin's (2008) model of geometric proof comprehension, Tauya's effort reveals that he had not interiorised matrix multiplication. According to Stylianides and Stylianides (2009) an argument is deemed to be valid if it is deductive and the premises logically imply the consequent statement. A true deductive argument once constructed offers complete and conclusive evidence about the truth of a mathematical statement. Hence, it becomes superfluous to look for further evidence about the truth-value of the mathematical statement (Weber & Mejia-Ramos, 2014). It can, therefore be, inferred from Tauya's use of a single example to resolve the proof task that he lacked a good grasp of these fundamental ideas about proving and proof in the area of Elementary Algebra.

C3: Argument produced does not represent the most general case

Table 1 shows that this category that emerged from inductive content analysis of students' was the most dominant among mathematics education postgraduates with 10 out of 30 responses. Next, the researcher now presents typical students' responses in this category and discusses these results within the perspective of Yang and Lin's model and other ideas about students' conceptions of mathematical proof. First Ticha's response is considered.

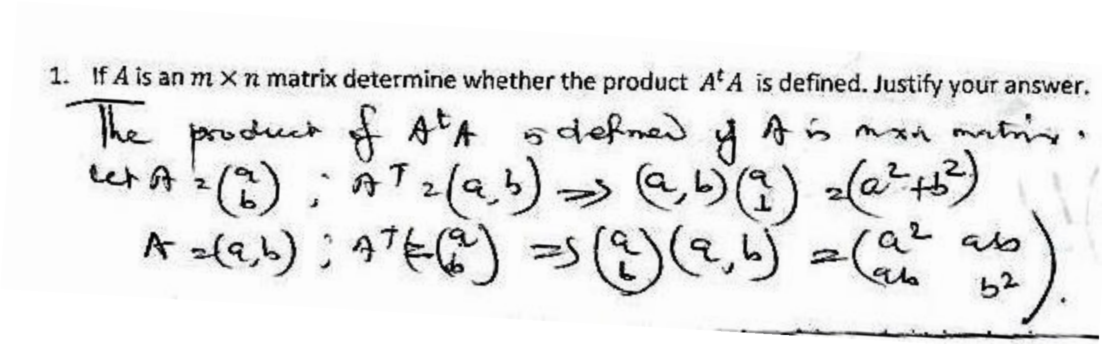

Figure 4. Ticha's written attempt to the task

Figure 3 shows that whilst Ticha produced his argument in terms of arbitrary mathematical objects by using the column matrix $A = \begin{pmatrix} a \\ b \end{pmatrix}$ whose transpose was correctly written as $A^t = (a \quad b)$, the argument cannot be considered to represent the general mxn matrix A. Similarly, a row matrix $A = (a \quad b)$ whose transpose $A^t = \begin{pmatrix} a \\ b \end{pmatrix}$ was also considered and the product A^tA=$\begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix}$ was then determined.

Hence, although in both cases Ticha's efforts involved manipulating arbitrary mathematical objects, the two cases cannot be deemed to represent the most general case of the product AB of two matrices A and B since Ticha's arguments involved column and row matrices. However, there was proper chaining of the elements (Azarello, 2007; Mejia-Ramos et al., 2012) as Ticha could carry out matrix multiplication correctly that led to the conclusion that "A^tA is defined if A is mxn matrix." It can be argued that traces of the encapsulation phase of proof comprehension were missing because Ticha could not conceive matrix multiplication in terms of the broader and more general case. Another example in this category by Mutaka is now presented and discussed.

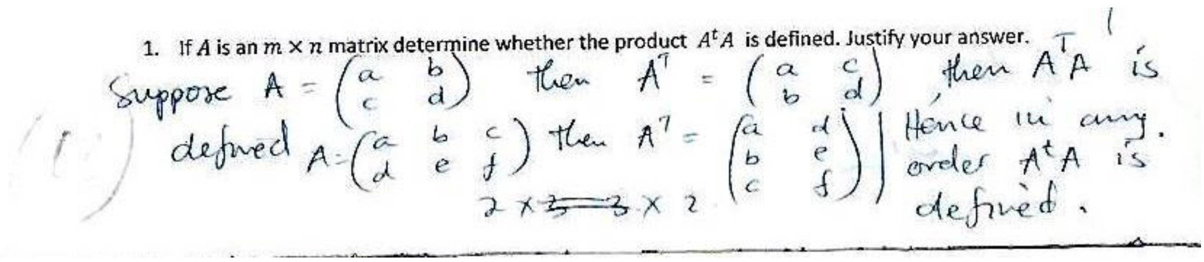

Figure 5. Mutaka's written effort to resolve the task

First, Mutaka considered the special case of a $2-$square matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and wrote the transpose of the matrix A correctly as $A^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. Mutaka's argument up to this stage indicated that the surface level of Yang and Lin's (2008) model of proof comprehension had been attained as shown by correct use of the definition of the idea of a transpose. However, chaining of the elements of A^t and A was not evident because after transposing the matrix A, Mutaka just wrote the conclusion that "then A^tA is defined." Second, Mutaka considered another special case of a $2x3$ matrix A with arbitrary entries and as before he managed to interchange rows and columns of A to get the transpose A^t. Similar to his proof behaviour in the first example discussed, there was no chaining of elements observed here that led to the conclusion stated as "Hence in any order A^tA is defined." In a similar fashion to Ticha's example the $2-$square and $2x3$ matrices used by Mutaka to support his conclusion that the product A^tA is defined for any mxn matrix A over a field of scalars K cannot be considered to be representative of the most general case of matrix multiplication. Hence, the encapsulation level (Yang & Lin, 2008) was not reached by Mutaka and Ticha. Next, the researcher focuses on informants' written responses in which the conclusion was either missing or wrongly formulated.

C4: Consequent statement missing or wrongly formulated

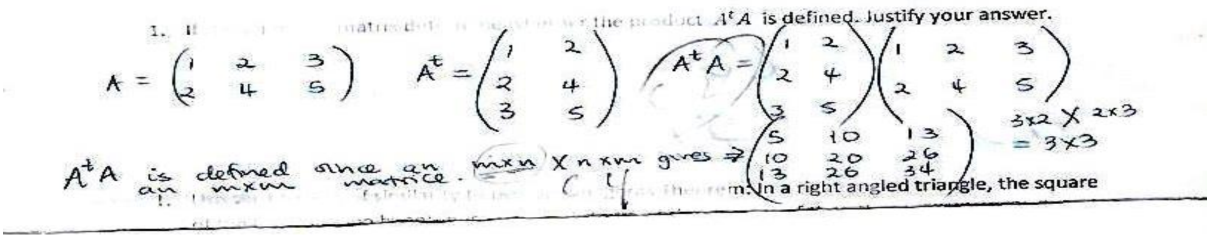

Figure 6. Mushai's written response to the task on matrices

Figure 6 shows that the categories formed from the analytic induction of data were not mutually exclusive because similar to efforts by Munya and Tauya, Mushai also used specific examples to validate the statement that $A^t A$ is defined. In addition to employing empirical verifications Mushai's arguments also revealed the following limitation. Mushai did not recognise that matrix multiplication is not commutative as he referred to A^t as a mxn matrix yet matrix A has been given as a mxn matrix. Furthermore, he identified A as nxm in stark contrast to the assertion that A was given as a mxn matrix. This chaotic proof behaviour led to the conclusion that $A^t A$ is "a mxm matrix," instead of an n −square matrix. Hence, whilst Mushai later on chained the elements correctly his failure earlier on to correctly identify the order of the matrix A led to the wrong conclusion. Another example in this category is now examined.

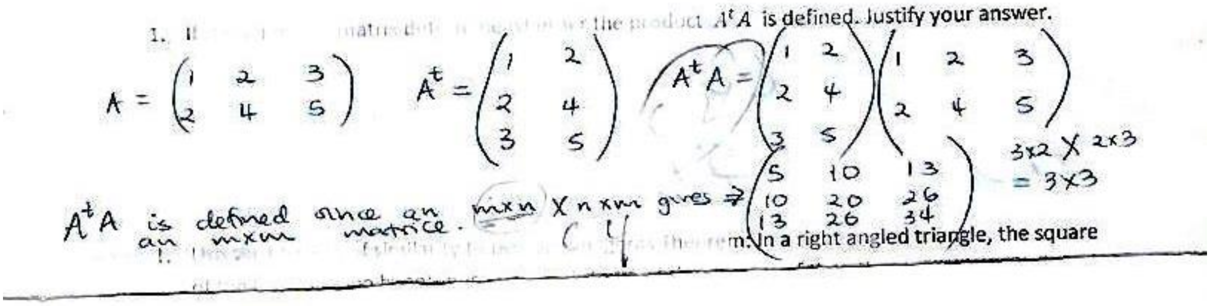

Figure 7. Maunja' written response to task on matrix multiplication

The product $A^t A$ denotes that the matrix A is post-multiplying the transpose matrix A^t. Hence, for compatibility of the multiplication operation the number of columns of A^t should be equal to the number of rows of A. Figure 7 illustrates that Maunja's argument contradicts to the assertion just stated concerning matrix multiplication. He wrote that "$A^t A$ will be defined … the no of columns in matrix A will always be equal to the no of rows in the 2^{nd} matrix." The second matrix mentioned by Maunja presumptively referred to the post-multiplying matrix which in this case should be the matrix A. Maunja's conclusion is a typical example of many such conclusions (7 out 30) drawn by postgraduate students that did not logically follow from the premises. Finally, the researcher focuses on a typical example of a flawed argument caused by an algebraic slip made by Mujuru.

C5: Argument has algebraic slips

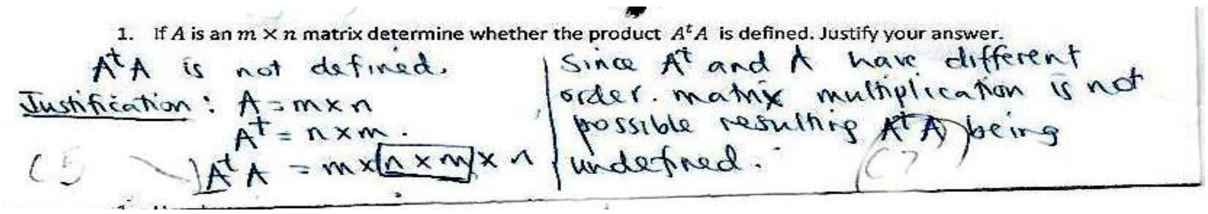

Figure 8. Mujuru's proof attempt to the task on matrices

Figure 8 shows that Mujuru started very well as she was able to write the transpose matrix A^t correctly as nxm matrix. However, her woes with the proof task manifested at the chaining stage (Yang & Lin, 2008) whereby her representation of matrix A^t as mxn led to the assertion that "$A^t A = mxn\, X\, mxn.$" This statement made it impossible for her to determine the product $A^t A$ and led to the conclusion that "A^tA is not defined." It can be observed also from the assertion "$A^tA = mxn\ X\ mxn$," that the two matrices A^t and A have the same order, which is a false assertion. Hence, the wrong conclusion drawn can be attributed to the algebraic slip made by Mujuru.

CONCLUSION

Postgraduate had not grasped the fundamental limitation that empirical verifications do not count as proofs. A case in point was the use of a single instantiation by Tanya. Furthermore, students produced arguments which were not typical of the most general. Although student teachers' efforts to justify that the product is defined were terms of arbitrary mathematical objects, those objects did not represent the most general case. For instance, a column matrix $A = \begin{pmatrix} a \\ b \end{pmatrix}$ was used to prove that the product A^tA exists. The cases used by postgraduate were not representative of the general matrix multiplication discussed in Section on multiplication of matrices. However, the use arbitrary objects was a huge step forward in current efforts to promote deductive argumentation among students in mathematics education. Lastly, postgraduate students' written responses revealed that the premises and the conclusion drawn were not logically connected. Further, in other cases the definition of the transpose of a matrix was not properly grasped as shown by proof behaviour such as referring to A^t as a mxn matrix when the matrix was stated as a mxn matrix. In addition, in some cases for the product, A^tA, the students focused on the number of columns of the matrix A instead of the number of rows since A was the post-multiplying matrix.

As concluding remarks, the researcher emphasizes that postgraduate students" flawed conceptions uncovered by this study have important implications for teacher preparation in Zimbabwe. Subject content mastery by students was fragile and hence the need for continuing professional development of in-service mathematics on subject content knowledge. The need to promote good grasp of concepts of Linear Algebra implies that mathematics educators and researchers need to find ways of ameliorating flawed conceptions of the concept of a mathematical proof. Furthermore, teacher preparation needs to include content and instructional strategies that foster and enhance prospective secondary mathematics teachers' explanatory role. Such content and strategies should also develop an appreciation of the function of a mathematical proof in justifying why a given mathematical assertion is true or false.

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