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STUDYING HOW PRIMARY-SCHOOL IN-SERVICE TEACHERS CONJECTURE AND PROVE: AN APPROACH FROM THE MATHEMATICAL COMMUNITY

Abstract

This paper studies how four primary-school in-service teachers develop the mathematical practices of conjecturing and proving in a working session on professional development while discussing questions that invoke these two mathematical practices. From the consideration of professional development as the legitimate peripheral participation in communities of practice, these teachers' practices (conjecturing and proving) have been analysed by using categories of activities originally given to characterise how a research mathematician develops these two mathematical practices. Our results show the significant presence of informal activities when the four participants conjecture, while few informal activities have been observed when they try to prove a result. In addition, the use of examples (an informal activity) is different in both practices since examples support the conjecturing process while constitute obstacles for the proving process. Finally, our findings are contrasted with other related studies and some suggestions that may derive from our work to enhance professional development are presented.

Keywords: Conjecturing, Proving, Primary-school in-service teachers, Professional development, Research mathematicians

Abstrak

Makalah ini mempelajari cara empat guru yang mengajar di SD dalam mengembangkan praktik melakukan konjektur dan membuktikan matematika dalam sesi kerja tentang pengembangan profesional sekaligus membahas pertanyaan yang menggunakan dua praktik matematika ini. Berdasarkan pertimbangan pengembangan profesional sebagai partisipasi periferal yang sah dalam komunitas praktik, praktik guru ini (melakukan konjektur dan membuktikan) telah dianalisis dengan kategori aktivitas yang awalnya ditentukan guna mencirikan cara seorang matematikawan penelitian mengembangkan dua praktik matematika ini. Hasil kami menunjukkan adanya aktivitas informal yang signifikan saat keempat partisipan melakukan konjektur, sementara sedikit aktivitas informal yang diamati saat mereka mencoba untuk membuktikan hasilnya. Selain itu, penggunaan contoh (aktivitas informal) berbeda di kedua praktik karena contoh mendukung proses melakukan konjektur, sekaligus merupakan hambatan dalam proses membuktikan. Pada akhirnya, temuan kami dibandingkan dengan studi terkait lainnya dan ditampilkan beberapa saran yang dapat berasal dari karya kami guna meningkatkan pengembangan profesional.

Kata kunci: Melakukan konjektur, Membuktikan, Guru yang mengajar SD, Pengembangan profesional, Matematikawan penelitian

How to Cite:

In mathematics education, research into mathematical practices is receiving increasing attention. This growth in interest is, on the one hand, motivated by studies from the philosophy of mathematics that focus on the processes of construction of mathematical knowledge (Lakatos, 1976; Tymoczko, 1998) and, on the other hand, by suggestions of a curricular nature that explicitly indicate the inclusion of mathematical practices as academic mathematical content (National Council of Teachers of Mathematics, 2000; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; RAND Mathematics Study Panel, 2003; U.K. Department of Education, 2014). Throughout this paper, *mathematical practices* (disciplinary practices in Rasmussen, Wawro, & Zandieh, 2015) are considered as being those mathematical activities developed by research mathematicians when building mathematical knowledge during their research. In this study, we focus on the mathematical practices of conjecturing and proving, for its distinguished role in the teaching and learning of problem solving (Chong, Shahrill, & Li, 2019) and in view of their relevance for the development of mathematical knowledge (Turrisi, 1997). Indeed, the existing literature brings to light the significance of these practices as a mean for deep learning in all the mathematical areas so that students' engagement in these activities can significantly influence their mathematical training, giving them a wide background for conceptual understanding (Stylianides & Ball, 2008; Watson, 1980).

Research into mathematical practices developed specifically by in-service teachers, which may give valuable information for the improvement of the teaching and learning of such practices, is a field that could be further explored. This research field may contribute, among others, towards the design of professional development activities, towards the professional noticing of mathematical practices, and consequently towards the promotion of students' understanding of these practices and towards their engagement therein. Several interesting contributions in this respect that focus on conjecturing and proving are found in: the research by Knuth (2002), which analyses teachers' conceptions of the role and nature of proof; that by Ko (2010), which makes a review of mathematics teachers' conceptions of proof and discusses possible implications for educational research; the paper by Melhuish, Thanheiser, and Guyot (2018) on professional noticing of the mathematical practices of generalising (conjecturing) and justifying (argumentation and proof); and the research by Lesseig (2016), which studies teachers' conjecturing, generalising and justifying behaviour when becoming involved in a classic number theory task that invokes these three mathematical practices. Another recent contribution on this topic by Astawa, Budayasa, and Juniati (2018) has studied the process of future mathematics teacher cognition in constructing mathematical conjecture.

This paper studies the mathematical practices of primary-school in-service teachers and, specifically, focuses on the way in which these teachers develop the mathematical practices of conjecturing and proving in a professional training context.

THEORETICAL BACKGROUND

Conjecturing and proving constitute two mathematical practices that have been extensively studied by the research community over the years. Consequently, many authors from various fields have referred to these practices and have assigned them different connotations. For this reason, this section begins by specifying certain terminology to frame our research. As mentioned earlier, the term *mathematical practice* is employed to refer to the mathematical activities developed by research mathematicians when they build mathematical knowledge during their research. In particular, the term *research mathematician* refers to those researchers who have a Ph.D. and have published research papers in mathematics. Furthermore, *conjecturing* and *proving* are the mathematical practices that generate conjectures and proofs respectively. Specifically, a conjecture is assumed to be a statement that can be true or false, appears reasonable, "has not been convincingly justified and yet it is not known to be contradicted by any examples, nor is it known to have any consequences which are false" (Mason, Burton, & Stacey, 1982, p. 58). Moreover, the definition of proof by Weber and Mejia-Ramos (2011) is adopted: "the socially sanctioned written product that results from mathematicians' attempts to justify why a conjecture is true" (p. 331).

In reviewing the literature, many studies can be encountered that advocate conjecturing and proving as two mathematical and very closely related practices which constitute two sides of the same coin. Several significant authors who support this consideration include: Pierce (Turrisi, 1997) and Lakatos (1976), from the philosophy of mathematics; and Polya (1954), Alibert and Thomas (2002), Boero and collaborators (Boero, 2007; Boero, Garuti, & Lemut, 2007), and Rasmussen et al. (2015), from mathematics education. All these authors highlight, from different points of view, that these two mathematical practices are interrelated and are essential for the construction of mathematical knowledge. For these reasons, the joint study of these two mathematical practices is considered in this paper.

In this study, the theoretical framework that is employed to analyse the data consists of several categories of activities that were identified to characterise how research mathematicians conjecture and prove (Authors, 2020). The use of that framework for this research is justified next. In this work, professional development is considered as the legitimate peripheral participation in communities of practice (Lave & Wenger, 1991). This theoretical perspective maintains that a teacher learns mathematics when he/she engages in research mathematicians' standard practices approaching to the way research mathematicians do it. For this reason, knowing how research mathematicians develop mathematical practices may be conducive to describing and explaining how in-service teachers develop them. In particular, a characterisation of a research mathematician's mathematical practices of conjecturing and proving (Authors, 2020) is employed in this paper to study how several primary-school in-service teachers conjecture and prove. Indeed, research mathematicians' mathematical practices are also a source of information to improve the teaching and learning of these practices at any academic level, since they inform about "what it is that we want students to learn and how instruction should be designed" (Weber & Dawkins, 2018, p. 70). Over the last decades, many other studies that try to gain an accurate understanding of how research mathematicians develop mathematical practices have been conducted. For instance, Burton (2004) proposes a model regarding how mathematicians learn mathematical practices through their own research, Ouvrier-Buffet (2015) and Author2 and other authors (2018) study the mathematicians' mathematical practice of defining and Weber (2008) researches into the mathematicians' practice of proving.

The characterisation of the research mathematician's mathematical practices of conjecturing and proving given by Authors (2020) is now described. This characterisation encompasses categories of activities that describe and explain how a research mathematician conjectures and proves. To be precise, Authors (2020) found five categories of conjecturing activities (C.H.a, C.H.b, C.H.c, C.V.a, C.V.b) and six categories of proving activities (P.H.a, P.H.b, P.V.a, P.V.b, P.V.c, P.V.d). The letter C in the code of certain categories refers to conjecturing and the letter P refers to proving. In their paper, Authors

(2020) consider the constructs of *horizontal mathematising* and *vertical mathematising* of Rasmussen, Zandieh, King, and Teppo (2005) to organise the mathematical activities (the categories) that are identified when conjecturing and proving. For this reason, the codes of the aforementioned categories also include either the letter H, if the category of activities is of a horizontal nature, or V, if the category is of a vertical nature. Rasmussen et al. (2005) use these two constructs (horizontal and vertical mathematising) to describe how certain students develop the mathematical practices of defining, symbolising and algorithmatising and thus characterise their so-called *advancing mathematical activity*. On the one hand, horizontal mathematising "is mainly related to initial or informal ways of reasoning" (Authors, 2020, p. 3) and, on the other hand, "vertical mathematising refers to those activities built on horizontal activities with the aim of creating new mathematical ideas or realities" (Authors, 2020, p. 3). A brief description is now offered of each of the categories of the theoretical framework. Firstly, the categories of activities related to the practice of conjecturing are described. Specifically, the three categories of a horizontal nature are defined first and later the two ones of a vertical nature.

C.H.a) *Detecting patterns*. This category of activities refers to the experimentations with mathematical objects (such as a square, a function, and a vectorial space) in relation to a certain characteristic or observable property. In order to be precise, we refer to such logical reasoning and informal activities with mathematical objects that give rise to the detection of a certain pattern in a concrete mathematical context.

C.H.b) *Testing conjectures*. This category includes those experimentations with specific mathematical objects carried out to check whether a conjecture does hold.

C.H.c) *Modifying statements*. This category refers to the experimentations with the components of an already existing conditional proposition (regardless of whether it has been proved or not, that is, either a proved proposition or a conjecture) that involve the modification of its hypotheses or conclusion. Specifically, this experimentation involves proposing possible changes on the hypotheses or thesis of a certain statement. The reasons behind a proposal of changes may be of a different nature. For instance, the finding of a counterexample for a conjecture may motivate the consideration of changes in parts of a statement and the study of which of those changes are possible.

C.V.a) *Formalising patterns*. This category refers to the generalisation and formalisation of a pattern observed when experimenting with mathematical objects in relation to a certain observable property. Specifically, a previously observed pattern is generalised to formulate a conjecture.

C.V.b) *Formalising modifications of statements*. This category refers to the formalisation of the modifications of the hypotheses or the conclusion of an already existing conditional proposition (regardless of whether it has been proved or not). Notice that this formalisation gives rise to a conjecture.

The six categories of activities related to the practice of proving are now described. Specifically, the two categories of a horizontal nature are first described and later the four ones of a vertical nature.

P.H.a) *Detecting techniques or tools within proofs*. This category involves a careful study and examination of the characteristics and steps of other proofs related to the proof to be constructed. In order to be precise, this category refers to the search for proof techniques or tools used in other proofs that may fit in well with the new proof.

P.H.b) *Detecting patterns in examples*. This category includes experimentations with mathematical objects that satisfy the hypotheses of a certain conjecture with the aim of detecting patterns that may be extended to the general setting where the conjecture to be proved is formulated.

P.V.a) *Selecting and applying proving methods.* This category involves the selection and application of proving methods: by contraposition, by contradiction, by induction, etc.

P.V.b) *Using proof techniques or tools found within other proofs.* This category involves the application and use of proof techniques or tools found in other proofs.

P.V.c) *Applying known results.* This category of activities appears when known results are applied to build chains of logical implications.

P.V.d) *Formalising findings with examples.* This category refers to the extension and formalisation of the calculations and experimentations with certain mathematical objects that satisfy the hypotheses of a given conjecture. Those patterns that were previously detected in examples are formalised, giving rise to part of the mathematical proof that is being constructed.

Since this paper focuses on the study of the way primary-school in-service teachers conjecture and prove, the research question that we tackle in this study is: how can we characterise the mathematical practices of conjecturing and proving of primary-school in-service teachers by using the categories of activities from the theoretical framework?

METHOD

Participants and Context

The participants of this study were four primary-school in-service teachers (whose students were between 6 and 12 years old), named Julia, Poppy, Ivy, and Rose (pseudonyms). Poppy and Ivy were expert teachers who had more than 20 years of experience, while Rose and Julia were newly qualified teachers. Each of the four had attained a three-year Bachelor’s degree in Primary Education and had completed a complementary course on the teaching and learning of mathematical problem-solving. It should be borne in mind that the Bachelor’s degree in Primary Education that these participants had studied included an approach to mathematical proofs (both numerical and geometrical) in its curriculum, although of a low-level.

These teachers participated in a professional training context where they worked together with educator-researchers from a Spanish University with the aim of improving their teaching of mathematics. This professional training context is the source of data of our research.

Data

The data of this study comes from the research described by Muñoz-Catalán (2009) in her Ph.D. thesis. While reading this dissertation for other purposes, one of the authors of the present paper informed the others that such dissertation included the transcripts of some conversations among four primary-school in-service teachers that could be interesting and useful for their research, since these conversations included these teachers' reasoning and reflections when constructing conjectures and proofs.

For this reason, we studied in detail the data included in that dissertation and, finally, analysed the transcript of a working session where the participants (the four in-service teachers described above) and the educator-researchers discussed two questions of a questionnaire (also designed by Muñoz-Catalán, 2009) on professional development. These two questions, laid out below (see Table 1), invoke the mathematical practices of conjecturing and proving.

[INSERT TABLE 1 HERE]

We notice here that, although most of the questions in the questionnaire (which included 30 questions) studied teachers' professional identities and their conceptions on the teaching and learning of mathematics, Muñoz-Catalán (2009) also took the opportunity to ask about professional knowledge and, specifically, about content knowledge (questions 8 and 11).

Data Analysis

A description of the process of analysis is now given. Firstly, each researcher individually distinguished events in the transcript linked either to the practice of conjecturing or to the practice of proving, according to the nature of each event. Subsequently, each researcher analysed the identified events by using the characteristics that describe each category of activities presented in the theoretical framework. Specifically, when an identified event was in line with the description of a category, that event was characterised by that category. In those cases where the identified event was related to (partially in line with) the characteristics of a certain category of activities, such an event was analysed independently and the slight differences between the event and the category were highlighted. We want to note here that we have not detected any event in the data related to the mathematical practices of conjecturing and proving that is not related (at least partially) to any category of the theoretical framework.

Finally, a contrast analysis was carried out between the assignments of events to categories conducted by each researcher. Those assignments that were common to the three researchers were accepted, and the non-common ones were discussed, in order to reach a consensus.

RESULTS AND DISCUSSION

This section is devoted to answering the research question of this inquiry and to discussing the findings. The classification of activities given in Authors (2020) is employed to describe and explain how four primary-school in-service teachers develop the mathematical practices of conjecturing and proving. Specifically, we show several relevant events identified in the transcript that are related to these two mathematical practices. This section is divided into four subsections. In the first two subsections, each of those events is characterised by using the categories of activities of the theoretical framework. In each event shown, those words that connect such an event with the category that characterises it have been underlined. In the third subsection, the slight differences found between the characteristics of some events and the descriptions of certain categories are highlighted and analysed. In this case, the words that connect each event with the category to which it is partially related have been underlined, although these events are not actually characterised by any category. In order to facilitate the reading of the results, the labels Q.8 or Q.11 have been assigned to most of the events we show, depending on whether the event is related to question 8 or question 11 (see Table 1). Moreover, the statement "The sum of a multiple of 2 and a multiple of 10 is a multiple of 10" shown in Q.8 will be referred to as C.8, while the statement "The sum of a multiple of 2 and a multiple of 10 is an even number" given in Q.11 will be referred to as C.11. In this paper, we assume that C.8 and C.11 are conjectures since, for the participants, these statements fulfil the conditions given in the definition of conjecture (see Theoretical Background section). Finally, in the fourth subsection, the findings of this study are discussed.

In this paper, our findings are reported through the separation of the activities of the participants that are linked to the mathematical practice of conjecturing from those that are related to the mathematical practice of proving and, specifically, by following the order of the categories presented in the Theoretical Background section.

How Primary-School In-Service Teachers Develop the Mathematical Practice of Conjecturing

Evidence is first provided of horizontal mathematising activities of the participants related to the practice of conjecturing.

C.H.a) *Detecting patterns.* This category of activities is identified when the participants experiment with specific numbers while carrying out arithmetic operations with them and, as a consequence of this experimentation, the participants detect a pattern.

A representative protocol of this category identified in the data is given below.

71. Ivy: No, I said I don't know how to prove it; the only thing I can say is that I have checked, <u>I have observed it, in the sense that I have seen</u>, I have sounded out it, <u>I have seen what happens</u> and later I have obtained a deduction from what I have observed, that an even number plus another even number is always even. (Q.11)

In this protocol, Ivy tries to answer Q.11. Specifically, she repeats a reasoning previously given in line 16 of the transcript, where she said that "I have written that I don't know how to prove it, I just know to check them and after observation and deduction from my observation, I say that an even number plus another even number is another even number". Although both reasonings (line 16 and line 71) are given in the process of proving of C.11, we have identified, when Ivy was testing C.11 ("I have checked"), that she detects a pattern in her calculations ("I have observed it, in the sense that I have seen […], I have seen what happens"). Notice that this detected pattern informs about a property that is different from the statement she was trying to prove (C.11). This is the main reason why we have linked this horizontal mathematical activity to the category *Detecting Patterns*. Specifically, from the expression "I have seen what happens" in line 71, we infer that Ivy has realised, after observing certain calculations, that something special or worthy of note may be taken into account. Finally, it can be noted that Ivy finishes her contribution by formalising the detected pattern in a conjecture. In this case, the formalisation of the pattern detected was essential for us to link this protocol to the mathematical practice of conjecturing since no data regarding her calculations is available.

There now follows certain protocols that show the appearance of the category *Testing conjectures* in the data.

C.H.b) *Testing conjectures.* This category of activities is identified when the participants check a previously given conjecture (C.8 or C.11) by using calculations with different numbers that satisfy the hypotheses of such a conjecture.

Four representative protocols of this category identified in the data are now shown. In this case, more examples are given to illustrate different consequences of the testing process.

1. Poppy: I first wrote down that I would need a mathematical proof because there are cases where it holds true and cases where it doesn't, but <u>because I was sounding out,</u> […], <u>then I said no, because the result of the sum 2+10 is not a multiple of 10</u> and then I said that <u>the moment it does not hold true for a case, a mathematical proof cannot be given.</u> […]. (Q.8)

35. Julia: Let's see, a multiple of 2 plus a multiple of 10, and now the 11th [she refers to Question 11] says that it has to be even and I find the fact of being even very logical but that it is multiple of 10 implies fulfilling many conditions, I will have to take 2 out as common factor. This is obvious because even numbers, by definition, are multiples of 2 but a multiple of 10 is more complicated because to be a multiple of 10, this written here in parentheses [$x + 5y$] will have to be 5 or a multiple of 5 and this is more complicated. <u>This fact holds true in some cases but not in all</u> [in the transparency this is shown as follows (see Figure 1)]. (Q.8)

[INSERT FIGURE 1 HERE]

98. Julia: Anyway, this case is too specific because it holds true in fewer cases than where it does not hold. Normally, it holds true for number 5 and number 10, but it does not hold for 6, 7, 8 nor for 9, that is, there are many cases where it does not hold true, double, compared with those where it holds. Then, I don't see either much need of more proof, this was seen as very clear, that there are proofs where something must be adapted because it is much more questionable, maybe in this case when one proves two million times, then you realise but since this case was so clear. (Q.8)

71. Ivy: […]; the only thing I can say is that I have checked, […] I have sounded out it, I have seen what happens and later I have obtained a deduction from what I have observed, […]. (Q.11)

In the first protocol, Poppy tries to answer Q.8. With this aim, she tests the conjecture C.8 with the numerical example offered as third possible answer to Q.8 (see the expression 2+10 in Table 1). Consequently, she finds a counterexample and rejects C.8.

In the second and third protocols, Julia studies under which additional conditions on the hypotheses of C.8, this conjecture could be true. First, it should be highlighted that she had previously noticed, at the beginning of the working session, that C.8 does not hold since she had found a counterexample by looking at the third possible answer to Q.8. In the first of these two protocols, Julia states that the fact of being a multiple of 10 (see the thesis of C.8) "implies fulfilling many conditions". For this reason, she tries to find which additional conditions on the hypotheses of C.8 would be necessary to ensure that the sum of a multiple of 2 and a multiple of 10 is a multiple of 10. In particular, she bases on the symbolic expressions used in the proof of C.11 to formulate a new conjecture, "[$x + 5y$] will have to be 5 or a multiple of 5" (for every natural number y), which provides details about the new additional conditions on the hypotheses of C.8. Moreover, she says that this new conjecture is true "in some cases but not in all", which has allowed us to infer that she has tested the new conjecture. In the third protocol, Julia explicitly mentions the numbers she has used to test the conjecture C.8 and specifies in which cases this conjecture holds (5 and 10) and in which cases it does not hold (6, 7, 8, 9). Here, it is significant that she highlights the need for more than one numerical example to reject the examined conjecture. We point out that such a type of behaviour has also been found in many other protocols of the transcript.

Finally, in the last example, we infer that Ivy tests the conjecture C.11 when she states that "I have checked, […] I have sounded out it".

A representative protocol that shows the appearance of the category *Modifying statements* in the data is now given.

C.H.c) *Modifying statements.* This category has been identified when the participants experiment with the components of an existing conditional proposition (C.8, C.11 or other propositions that appear in the data), by modifying its hypotheses or conclusion.

35. Julia: […] implies fulfilling many conditions, […] multiple of 10 is more complicated because to be a multiple of 10, this written here in parentheses [x + 5y] will have to be 5 or a multiple of 5 and this is more complicated. This fact holds true in some cases but not in all [in the transparency this is shown as follows (see Figure 1)]. (Q.8)

In this protocol, already shown and described above, Julia proposes an extra condition on the hypotheses of C.8 that would guarantee that a new, although weaker, result would be true (if x + 5y is multiple of 5 then 2x + 10y is multiple of 10). To be precise, she suggests that the sum of a multiple of 2 ($2x$) and a multiple of 10 ($2 \cdot 5y$) would be a multiple of 10 in the case that $x + 5y$ were multiple of 5. This last condition, which appears in the proof process of C.8, is an extra condition on the hypotheses of C.8. We feel that Julia does not formalise such a possible modification on the hypotheses of C.8 since she may not have realised that she is carrying out such a modification. That is, while she is trying to prove C.8, she realises that the conclusion of such a conjecture is too strong in the light of the written algebraic expressions she is working with. However, she may not be aware that this new condition considered in the proof could imply that she has proved a new result. More data would be needed to establish stronger conclusions on this matter.

The evidences of vertical mathematising activities of the participants linked to the practice of conjecturing are laid out below.

C.V.a) *Formalising patterns.* This category has been identified when the participants of this study generalise and formalise a certain previously detected pattern.

71. Ivy: […] I have obtained a deduction from what I have observed, that an even number plus another even number is always even. (Q.11)

In this case, it can be observed that Ivy has formalised the pattern detected while testing conjecture C.11 by giving a new conjecture (see an exhaustive description of this protocol at the beginning of this subsection). Figure 2 shows the new conjecture.

[INSERT FIGURE 2 HERE]

We want to note that this last new conjecture has a very similar structure to conjecture C.11. This may be due to the fact that Ivy detects the referred pattern, which is later formalised, when she is testing conjecture C.11. However, there is insufficient data to assert that the construction of this new conjecture (see Figure 2) is based directly on the modification of the statement of C.11. For this reason, this protocol has not been included as empirical evidence of the category *Modifying statements.*

C.V.b) *Formalising modifications of statements.* This category is identified when the participants formalise, in a new conjecture, the modifications they have previously considered on the hypotheses or conclusion of an existing statement.

122. Ivy: Sometimes yes, and sometimes no; I have not stopped to look at it, what happens is that an odd number plus another odd one is even, but it may be any even number, it does not have to be just the even-number multiples of 8.

This protocol is part of the discussion among the participants about the veracity of a conjecture posed by an educator-researcher during the analysed session, which states “the sum of a multiple of 5 and a multiple of 3 is a multiple of 8”. In this protocol, Ivy gives a new conjecture, which we call C’, that claims that “the sum of two odd numbers is an even number that is not necessarily a multiple of 8”. This new conjecture arises from the following process: in line 116, the educator-researcher deduces that Ivy, in her previous appearances, is asserting that “odd plus odd is not even”. Later, in line 117, Ivy denies such an assertion and gives the new conjecture C’ by modifying the previous assertion deduced by the educator-researcher. It can be observed that Ivy creates C’ in two almost simultaneous steps: the first part of the conjecture (“the sum of two odd numbers is an even number”) appears when Ivy modifies the educator-researcher’s assertion (“odd plus odd is not even”) by referring to a mathematical property (“odd plus odd is even”) that she already knows. Furthermore, the second part of the conjecture (“that is not necessarily a multiple of 8”) is closely related to the conclusion of the conjecture posed by the educator-researcher that she is originally trying to prove (“the sum of a multiple of 5 and a multiple of 3 is a multiple of 8”).

How Primary-School In-Service Teachers Develop the Mathematical Practice of Proving

Evidence of horizontal mathematising activities of the participants of this study related to the practice of proving is first provided.

P.H.a) *Detecting techniques or tools within proofs.* This category is identified when one of the participants reflects on the steps and characteristics of one existing proof with the aim of finding techniques or tools that may fit in well with the construction of a new proof.

44. Julia: Look, both statements are the same, the sum of a number multiple of 2 and another number multiple of 10; a multiple of 2 is 2x and a multiple of 10 is 10y; the sum of both numbers [writing down the sign + between both algebraic expressions (see Figure 3)]. Then what I have done is to take out 2 as common factor, so yes it is an even number since 2 times anything is always even, because an even number, by definition, is always going to be multiple of 2. And the other has the same statement, it says: the sum of a number multiple of 2 and another number multiple of 10, that is similar to this one, has to result in a multiple of 10. Therefore, for this [pointing out the expression $2(x + 5y)$ in the transparency (see Figure 3)] to be a multiple of 10 what I have thought is that if 10 is equal to $2 \cdot 5$ then $x + 5y$ has to be equal to 5 or multiple of 5. (Q.8)

[INSERT FIGURE 3 HERE]

At the beginning of this protocol, Julia describes the main steps that she has followed to prove C.11. She subsequently carefully reflects on these steps with the aim of finding techniques that may fit in well with the proof of C.8. To be precise, it can be observed at the end of the protocol that she tries to apply the scheme of the proof of C.11 to prove C.8 but does not succeed. This behaviour is highlighted again in P.V.b.

The evidences of vertical mathematising activities of the participants of this study linked to the practice of proving are laid out below.

P.V.b) *Using proof techniques found within other proofs.* This category has been identified when one of the participants uses techniques of one existing proof in order to prove a new result.

44. Julia: Look, both statements are the same, […]. And the other has the same statement, it says: the sum of a number multiple of 2 and another number multiple of 10, that is similar to this one, has to result in a multiple of 10. Therefore, for this [pointing out the expression $2(x + 5y)$ in the transparency (see Figure 3)] to be a multiple of 10 what I have thought is that if 10 is equal to $2 \cdot 5$ then $x + 5y$ has to be equal to 5 or multiple of 5. (Q.8)

In this protocol, Julia applies techniques found in the proof of C.11 to prove C.8. To be precise, she decomposes numbers into prime factors and takes out common factors. Furthermore, we have noticed that she is aware of what should happen to make this more complicated case (the proof of C.8) work. However, we feel that the complexity of formalising such ideas (the fact that $x + 5y$ is a multiple of 5) makes that, as mentioned earlier in this paper, she does not raise any new conjecture related to them.

Certain protocols can now be presented that show the appearance of the category *Applying known results* in the data.

P.V.c) *Applying known results.* This category has been identified when the participants apply certain properties or results to build chains of logical implications.

19. Julia: I have considered that an even number by definition is a multiple of 2 and, therefore $2x + 2 \cdot 5y = 2(x + 5y)$; then it is multiple of 2. (Q.11)

24. Rose: And the sum of two even numbers is always an even number. (Q.11)

In the first example, Julia applies the Fundamental Theorem of Arithmetic that states “every natural number can be written as a product of prime numbers”, together with the property of taking out common factors. In the second example, Rose applies a known result to finish a chain of logical implications that proves the conjecture C.11. To be precise, Rose firstly applies, on line 20, the Fundamental Theorem of Arithmetic to decompose 2 and 10 into the product of prime factors. She explicitly states there that “I have written down $2x +$, I have decomposed number 10 as $5 \cdot 2$ and then, I have written down that $2x + 5 \cdot 2y$ is equal to another number, equal to z, […]”. Subsequently, instead of taking out common factors in the same way as carried out Julia (see the preceding protocol), Rose directly applies a known result, “the sum of two even numbers is an even number”, to the algebraic expression $2x + 5 \cdot 2y$ to conclude that C.11 is true. It is important to notice here that several of the techniques or tools that the participants apply in this context of divisibility to answer the questions (Q.8 and Q.11) coincide with the application of classic theorems (such as the Fundamental Theorem of Arithmetic) or properties that are often used in this branch of mathematics.

Finally, it should be emphasised that certain categories of activities of the theoretical framework have not been identified in the data: P.H.b, P.V.a and P.V.d. However, related to P.H.b and P.V.d, we have found that at least one participant of this study is aware that the educator-researcher behaves according to the description of such categories when carrying out proving activities. The following example illustrates this.

133. Julia: This proof is in reverse. It starts from a specific case, you have not done it in the same way as the other proof.

In this protocol, Julia realises that the educator-researcher draws from particular cases (specific examples) to construct a proof (see Figure 4), which is a different method compared to that which the participants had carried out before.

[INSERT FIGURE 4 HERE]

More on Proving

There is a group of events identified in the transcript which, while informing about how the participants prove, have not been characterised by any category, since none of these events fulfil all the characteristics of a category. However, it is interesting to note that each of these events is partially related to some category of the theoretical framework, since there always may be found some similarities and also some differences between the characteristic of an event and the description of the related category. For this reason, these events cannot be directly characterised by the related category. For instance, the following protocol is closely related with the P.H.a category, although its characteristics are slightly different to the description of this category.

17. Poppy: I was also testing [as Ivy did in line 16] but, <u>maybe with an equation I can do something,</u> and I also did it […]. (Q.11)

In this protocol, we observe that Poppy is considering the technique of using “equations” (a word that we think she uses as a synonym of “algebraic expression”) to prove statements related to divisibility in natural numbers: a topic studied in her Bachelor’s degree. In particular, this protocol shows how primary-school in-service teachers may sometimes consider proof techniques that they already know while trying to prove certain new mathematical statements. It should be borne in mind that Ivy has not been observed as finding the referred proof technique in another proof, as the P.H.a category considers. For this reason, this event has not been characterised by this category.

Other examples are shown in the two protocols below. In this case, these protocols are closely related with the P.V.b category, although their characteristics are slightly different to those in the description of this category.

19. Julia: I have considered that an even number by definition is a multiple of 2 and, therefore $\underline{2x + 2 \cdot 5y = 2(x + 5y)}$; then it is multiple of 2. (Q.11)

66. Poppy: Yes, I have said: $2n + 10n$, but later I said: well, <u>number 10 can be decomposed; then I wrote 5 to take out the common factor</u>. (Q.11)

In these two protocols, the participants use the techniques of decomposing numbers into prime factors or of taking out common factors when proving certain results. Notice that both techniques are widely used in divisibility. It should also be emphasised that, in these two protocols, not only have the two techniques described above been observed but also the technique of translating a statement into symbolic language (mathematical symbols). This technique encompasses the translation of the hypotheses or conclusion of a statement into symbolic language in the proof process of such a statement. For instance, on line 19, Julia considers the definition of an even number and writes down its translation (the algebraic expression $2x$) with the aim of proving a statement. Specifically, this technique has been found several times in the data when the participants try to prove conjectures (C.8, C.11, or other conjectures that the educator-researchers pose), and translate the statements of such conjectures into mathematical symbols.

In addition, these two protocols show that primary-school in-service teachers sometimes employ proof techniques that they already know while trying to prove certain new mathematical statements. As before, it should be borne in mind that Julia and Poppy have not been observed to have found the referred proof techniques in another existing proof, as the P.V.b category demands.

On line 19, another interesting event may be highlighted. Specifically, the event "I have considered that an even number by definition is a multiple of 2 [...] $2(x + 5y)$; then it is multiple of 2." is closely related with the P.V.c category, although the characteristics of this event are slightly different to the description of this category of activities. In this event, Julia firstly takes into account the definition of an even number to be aware of what she has to obtain in order to prove C.11. Subsequently, she writes down algebraic expressions related to the hypotheses of C.11 "$2x + 2 \cdot 5y = 2(x + 5y)$", and finally she applies the definition of even number in order to conclude that C.11 is true. This event shows that primary-school in-service teachers may sometimes apply definitions or similar statements (axioms) when building chains of logical implications. It should be noted that, in this event, Julia does not apply a known result but a known definition, which allows her to construct a chain of logical implications in a similar way to that in which the P.V.c category describes.

Some reflections on results

In this paper, the mathematical practices of conjecturing and proving of four primary-school in-service teachers have been characterised. The various categories of activities of the theoretical framework identified in the data have helped us to describe and explain how these teachers develop these two mathematical practices. Moreover, we have highlighted that some events identified in the data are related to certain categories of activities, although those events present some characteristics that differ from the characteristics of the categories to which they are related. Thus, the findings reported in this work inform that not all the mathematical activities developed by primary-school in-service teachers when conjecturing and proving may be exhaustively explained by the categories of activities defined in Authors' (2020) study. We also note that three categories of activities of the theoretical framework have not been identified in the data (P.H.b, P.V.a, P.V.d) and that these three categories are linked to the mathematical practice of proving.

Furthermore, we also highlight that the majority of the events identified in the data and subsequently categorised with the theoretical framework are of a horizontal nature, that is, they mainly include informal ways of reasoning. In fact, the main behaviour that guides the participants' practice of conjecturing is that of the use of examples. In particular, they use mathematical objects that satisfy the hypotheses of certain statements to test said statements, which help these teachers to reject such statements or convince themselves of its truth. As a consequence of the testing process, these teachers sometimes observe regularities that motivate the appearance of new conjectures. In previous studies conducted with pre and in-service teachers (see, for instance, Knuth, 2002; Martin & Harel, 1989), the role of empirical evidences in obtaining conviction regarding the truth of a statement is also highlighted. In particular, Knuth (2002) indicates that teachers "reach a stronger level of conviction" (p. 401) by testing with specific mathematical objects. In our work, we have even noticed that the participants sometimes show confusion between the process of testing with specific mathematical objects and the mathematical proof. Nevertheless, what we can conclude is that the participants of the present study grant the experimentation with mathematical objects an important role in the development of the mathematical practices of conjecturing and proving.

The literature on students' difficulties with proofs has documented a variety of persistent misconceptions of the students (Stylianides & Stylianides, 2018). One such misconception is that *a single example is insufficient to reject an assertion or mathematical property*, and another is the fact that *the verification of a property in some particular cases is enough to guarantee its general validity*. Our results with primary-school in-service teachers show that these two misconceptions persist with certain teachers. This fact reveals a *pernicious circle* where students maintain these two misconceptions from school or high school until they become in-service teachers, and they then continue transmitting the same misconceptions to their own students. We suggest that giving opportunities for professional development on these mathematical practices may help to break this vicious circle. For instance, the results of this study underline the need for the promotion of aspects of a more formal nature of the practice of conjecturing (vertical mathematising activities) and aspects of the practice of proving such as the use of examples that guide the proving process but do not constitute a proof. From our learning approach, we state that learning in the processes of professional development aims, through the peripheral participation, to approach the way in-service teachers develop mathematical practices to the way mathematicians' community of practice develops such practices (Lave & Wenger, 1991), always taking into consideration that there are differences between both contexts (Weber, Inglis, & Mejia-Ramos, 2014).

The literature has also documented that primary-school in-service teachers often face similar difficulties that students and pre-service teachers face with many other mathematical contents, not only with proofs (see, for instance, Ubuz & Yayan, 2010). For this reason, we agree with Ubuz and Yayan (2010) when stating that "an important step to improving subject matter knowledge should be better subject matter preparation for primary teachers" (p. 799). We also believe that addressing difficulties in the content knowledge of primary-school teachers would shed light on where to focus to get teachers to acquire essential content knowledge to teach in the various mathematical domains. In this way, studying teachers' activities when they try either to prove mathematical assertions or to formulate conjectures is the first step and exploring the reasons behind the difficulties found when conjecturing and proving would be an interesting topic for further research.

We also find it interesting to make a comment on line 71 of the transcript (see C.H.a category above). In this protocol, where Ivy explains how she poses the conjecture: "the sum of two even numbers is an even number", we identify a chain of categories of activities (*chain of progressive mathematisations* in Rasmussen et al., 2005) that explicitly shows how the theoretical framework is able to explain the way primary-school in-service teachers develop the mathematical practice of conjecturing. This chain begins when Ivy tests C.11 (C.H.b) and this chain continues when she later observes a pattern in her calculations (C.H.a) that it is not immediately deducible from C.11. Moreover, it can be seen that this observed pattern is formalised in a new conjecture: "the sum of two even numbers is an even number" (C.V.a). As mentioned in a previous subsection, the fact that the new statement has a similar structure to that of C.11 could inform us that Ivy has based the construction of the new conjecture on the statement of C.11. However, more data would be required to conclude that the C.H.c or C.V.b categories appear in line 71. The chain of categories found in this protocol reveals possible interconnections among the categories of activities of the theoretical framework and shows how vertical activities build on horizontal activities. Furthermore, this chain, which shows conjecturing activities while a participant is proving a statement (C.11), is a clear example of what Lesseig (2016) calls a "cycle of empirical exploration, conjecturing, generalizing and justifying" (p. 18). On line 71, Ivy reasons "through examples to generate additional conjectures and generalizations" (Lesseig, 2016, p. 18).

CONCLUSION

Our work characterises how several primary-school in-service teachers develop the mathematical practices of conjecturing and proving. Melhuish et al. (2018) have recently affirmed that a "teacher must be able to notice mathematical reasoning forms such as *justifying* and *generalizing*" (p. 2) (herein named proving and conjecturing). In this regard, our results highlight aspects of these practices on which professional development must focus so that in-service teachers are able to notice different forms of mathematical reasoning in order to foster mathematical practices in classrooms.

Furthermore, other mathematical tasks that invoke the same mathematical practices (conjecturing and proving), although from different mathematical fields (such as analysis and geometry), should be considered to complement this exploratory study. We also maintain that the mathematical practices of conjecturing and proving of more primary-school in-service teachers should be studied to achieve findings of a more representative nature. We think that the fact that the data of this study is limited may have influenced that the theoretical framework has been really useful to describe how these primary-school in-service teachers conjecture and prove. Outside the scope of the theoretical framework, we have only observed that the participants sometimes consider and employ proof techniques that they already know and that sometimes apply definitions or similar statements (axioms) when building chains of logical implications. In any case, we believe that these last findings are a starting point to broaden the theoretical framework defined by Authors (2020), although for this purpose more research mathematicians should be studied, since the categories of activities were generated by this population. Indeed, we hypothesise that more research on this topic could show new mathematical activities when conjecturing and proving that motivate the appearance of new categories of activities.

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FIGURE CAPTIONS

Figure 1. Hand-written note included in line 35 in the transcript

Figure 2. Hand-written note found in the data that shows the formalised pattern. Notice that the term "par" in the note means "even"

Figure 3. Julia's hand-written note that shows her steps to prove C.8

Figure 4. An educator-researcher's hand-written note found in line 132

TABLE CAPTIONS

Table 1. Questions 8 and 11 from a questionnaire on professional development (Muñoz-Catalán, 2009, vol. I, p. 232)

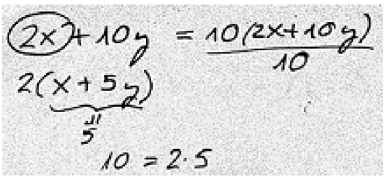

Figure 1

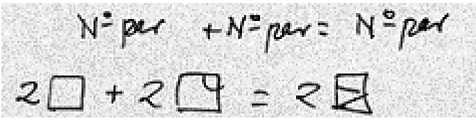

Figure 2

Figure 3

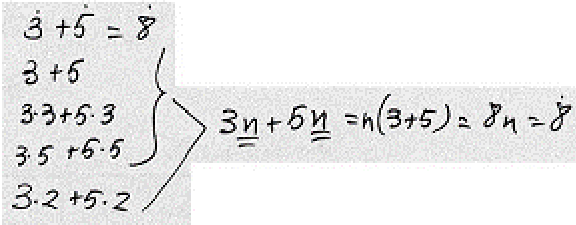

Figure 4

Table 1

QUESTION 8: Is the following statement true: “The sum of a multiple of 2 and a multiple of 10 is a multiple of 10”?

- ☐ Yes, because 20+40 is a multiple of 10.
- ☐ Yes, because it holds true for the following examples: 10+10, 20+10, 50+20.
- ☐ No, because the result of the sum 2+10 is not a multiple of 10.
- ☐ I would need a mathematical proof because there are cases where it holds true and cases where it does not.

QUESTION 11: Prove whether the following statement is true: “The sum of a multiple of 2 and a multiple of 10 is an even number”.